A·L·G·E·B·R·A a vital ingredient

by W. W. Sawyer

The last twenty years or so have seen a marked decline in the teaching of elementary algebra. This is to be regretted. I do not for a moment wish to see pupils slaving away at soul-destroying drill on algebraic routines in the mistaken belief that this will improve examination prospects. It seems worth while to consider briefly the uses of algebra when the subject is approached properly.

One outstanding use of algebra is to make the learning of arithmetic more interesting, and enable it to give the same kind of creative satisfaction that children may get in art from making a drawing or in English from writing a story.

I demonstrated in *Vision in Elementary Mathematics* that algebra can be made clear and interesting to quite young children by suitable diagrams and — a more severe test — I have also used this approach successfully with older pupils of limited mentality.

Mathematics through Vision

As *Vision in Elementary Mathematics* has been out of print for quite a number of years, it may be useful to sketch the method described in it. We begin by presenting a well-known trick and asking each child to do what the trick asks for.

Think of a number, add 3 to it, double the result, subtract 4, divide by 2, take away the number you first thought of.

We then ask each child in turn to say what number came at the end of this procedure. The answers ought to be "One", "One", "One"... but it does not always work out like that.

Once I was doing this in a demonstration to a large number of American teachers with a Grade 5 class (ten-year-olds), and not a single child gave the answer, one. I asked each child to show what he or she had done, and discovered beliefs such as $2 \times 7 = 16$. After these mistakes had been ironed out, the whole class agreed on the answer, one. The trick proved quite a good way of testing and encouraging accuracy.

We next raise the question, why does the answer, one, always come? Or indeed, must it always come? May there be some awkward number that leads to another answer? Can we be sure that our trick will always work?

At this stage a way of visualizing the trick is introduced. We suppose the person, who is thinking of a number, puts that number of stones into a bag, so we have our picture;–

The number thought of

Then we ask the class how we are to visualize the next step, "add 3". A child may suggest putting 3 more stones into the bag. Mathematically this is sound enough but it is not very helpful as a visual aid. We may suggest that the stones be shown outside the bag.

Add 3

A real discussion arises when the class is asked to illustrate the next step, "double". Should it be 2 bags and 3 stones, or 1 bag and 6 stones, or 2 bags and 6 stones? All of these may well be suggested. It is up to the class to decide which is right, by argument or by trying what happens with particular numbers. Eventually there should be agreement on the picture;–

Double

The next step, "take away 4", usually causes no trouble. We remove 4 stones.

Take away 4

There may well be discussion on the next step, "divide by 2". Experiments with particular numbers, or considering the fair sharing out of the bags and stones between two claimants should in time lead to

Divide by 2

Finally, take away the number you first thought of. "Where is that number?" usually brings a chorus of "In the bag." We remove the bag.

Take away the number you first thought of

Here we have a proof that the answer will always be 1. We still do not know how many stones were put into the bag. We have carried out addition, subtraction, multiplication and division with an unknown number.

The children then proceed to invent their own tricks with the help of drawings of bags and stones. They test the correctness of their tricks by trying them on their classmates. In the course of this some large numbers may appear. No one wants to draw 10 bags and 128 stones, and an abbreviation becomes appropriate. We indicate this collection as

10  + 128.

After some work with this notation I am overcome by fatigue; I cannot be bothered to draw the bag properly, and I start to leave off the top and bottom of the bag. The class is asked what this simplified symbol looks like, and they usually reply, "the letter x". From then on we use x as a convenient way of indicating a bag, representing any number you care to choose. My 9 and 10-year-olds were quite unaware that they were doing algebra. They only realised this when older brothers and sisters said to them, "You have no right to be doing this. It is Grade 9 work, not Grade 5."

The Element of Surprise

It is slightly surprising that every member of the class can start with a different number and all of them end up with the same answer; the interest of the trick lies in this fact. Now arithmetic abounds in such mild surprises, which children enjoy spotting, and which it is the business of algebra to explain. These will come to light if we are careful not to set arithmetical exercises at random, but rather in a sequence chosen so that some pattern can be seen in the answers. For example, children may be asked to find the numbers 5×5, 6×4, 7×3, 8×2, 9×1, 10×0 and examine the answers, which are 25, 24, 21, 16, 9, 0. Pupils may notice that the downward steps are 1, 3, 5, 7, 9, the odd numbers in order. We may go on to ask if there is anything interesting to observe in the differences between these numbers and 25. It is apparent that all the differences are perfect squares.

$$\begin{aligned}
5 \times 5 &= 25 = 25 - 0 = 5^2 - 0^2.\\
6 \times 4 &= 24 = 25 - 1 = 5^2 - 1^2.\\
7 \times 3 &= 21 = 25 - 4 = 5^2 - 2^2.\\
8 \times 2 &= 16 = 25 - 9 = 5^2 - 3^2.\\
9 \times 1 &= 9 = 25 - 16 = 5^2 - 4^2.\\
10 \times 0 &= 0 = 25 - 25 = 5^2 - 5^2.
\end{aligned}$$

Surely this cannot be an accident. How can we picture what is happening? We take a particular result, say $7 \times 3 = 5^2 - 2^2$. How are we to picture 25, the square of 5? The word square suggests that we think of 5 rows of 5 objects. We are to subtract from this the square of 2. This suggests the following diagram, in which the shaded objects are supposed to have been removed.

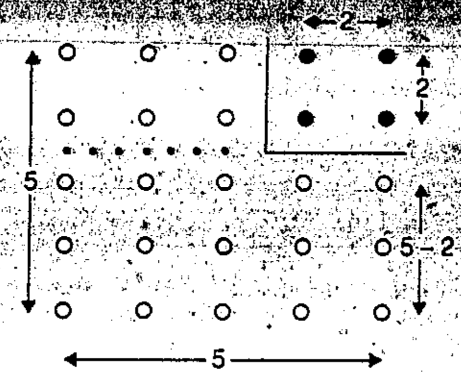

Fig. A

The number of objects remaining equals 7×3; that is to say, it should be possible to arrange them in 3 rows of 7. How is this to be done? Most classes will tell you quite quickly that you should slide the 6 objects above the dotted line to the position shown here;—

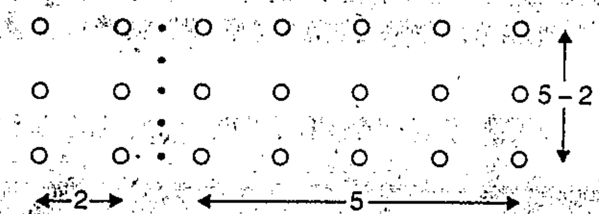

Fig. B

Incidentally, this procedure shows how the 7 and the 3 come from the 5 and 2 in Figure A; 7 is 5 + 2 and 3 is 5 − 2. Our result could be written $5^2 - 2^2 = 5 + 2$ multiplied by $5 - 2$. Now there is nothing special about the numbers 5 and 2 here. We could replace 2 by any number we liked and 5 by any number larger than that then draw diagrams similar to Figures A and B, and write the resulting equation.

This would bring us to a well known maxim in algebra — the difference of two squares is the sum multiplied by the difference, or in symbols

$$a^2 - b^2 = (a + b)(a - b).$$

Whether we would find it desirable to carry our discussion to this degree of formalism would depend on the age and the composition of the class.

I hope the above samples will be sufficient to justify my earlier statement that a major function of algebra can be to enliven the teaching or arithmetic in elementary schools.

We now look at other aspects of algebra.

Other Roles for Algebra

For pupils of average ability, there are obvious practical applications. Being able to use a formula intelligently and correctly is often the first step in technical training or the key to many branches of science.

For the very strong mathematician or scientist, facility with algebraic manipulation is indispensable. It does not rank so high as mathematical originality, ingenuity or insight, yet it is still important. In calculus, for instance, algebra perpetually occurs incidentally. A person who has to stop and wrestle with the algebra on each occasion is like someone who is trying to understand a book and is continually distracted by having to spell out individual words. It should be possible to deal with the algebra automatically and almost unconsciously.

For about ten years now I have had three or four able and interested youngsters coming to our home on Saturday mornings. They may join the group at the age of 13 or 14 and stop coming when they leave the sixth form. My experience with them is that they are capable of work years ahead of anything in the school curriculum, but that they become almost paralyzed when they have to manipulate symbols. Years pass before this weakness is even partially overcome.

At present our custom is to spend about half-an-hour at the end of the morning on algebra. Each member of the group chooses a question from the Miscellaneous Examples VI at the end of Hall and Knight's Elementary Algebra. We each work at the example and then discuss not only the problem itself but also how such a question might be composed and whether it suggests other results.

For instance, recently we did question 151, — a number is written with two digits, and it equals 6 times the sum of its digits; prove that, if the order of the digits is reversed, we obtain a number equal to 5 times the sum of the digits. The algebra is very simple. If the number is $10a + b$, both the data and the results to be proved are expressed by the same equation, $4a = 5b$.

The question arises — why 6 and 5? Could we have a similar question with other numbers? To answer this we consider the number being n times the sum of its digits. We then find the reversed number is $11 - n$ times the sum of its digits, so we could, if we liked, replace 6 and 5 by, say, 9 and 2. The statements about the number and the reversed number would then be expressed by equations (1) and (2).

$$10a + b = 9(a + b) \ldots (1)$$

$$a + 10b = 2(a + b) \ldots (2).$$

If we add these, we get equation (3)

$$11a + 11b = 11(a + b) \ldots (3)$$

This suggests an odd little proof. Since equation (3) is automatically true, if we are given equation (1) we can prove equation (2) by subtracting equation (1) from equation (3).

In the early days of "Modern Maths", the most successful teachers used SMP or some other new book, but reserved a certain amount of time for work on traditional algebra. I would like to see a revival of this practice. The last thing I would wish to see is drill on algebraic routines being forced on pupils who found them meaningless. Indeed I have long advocated that the less academic students should have their mathematics embedded in practical work with actual materials, which they find much easier to understand. (1, 2.)

But equally we should make sure that our future mathematicians and scientists are not denied the tools of their trade. I would be quite happy if a number of the older textbooks were kept in the classroom, so that those who finished the regular stint of exercises ahead of the rest of the class could choose some traditional problem and work at it individually or in small groups.

References

1. Srawley, L. G. and Sawyer, W. W. (1950) *Designing and Making*. L. G. Srawley and W. W. Sawyer. Basil Blackwell.
2. Sawyer, W. W. (1980) Two Avenues to Advance in Mathematical Education. *Mathematics Teaching*, pp. 22–26.

A Fistful of Digits

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This is the name of a problem posed in the *Open University* booklet *Calculators in the Primary School* for course PM537, page 37, question 5.

The Problem

23×45, 315×24, and 341×52 are products using the same five digits. Which arrangement of these five digits gives the greatest product?

Following an investigation with a calculator, the answer reached seemed to demand further explanation. The argument which follows does not invite the use of a calculator, but it would not have been put together unless the calculator had been available in the first instance.

Lemma

x and y are variables with a constant sum.

The smaller their difference, the greater their product.

Proof

Let $x + y = 2k$.

If $x = k - e$, then $y = k + e$.

So $xy = k^2 - e^2$.

Since k is constant, xy is greater when $|e|$ is smaller, and $|e| = \frac{1}{2}|y - x|$.

Solution of the Problem

Products > 20,000 are of four types:

(*i*) $4 \times 5abc$,
(*ii*) $5 \times 4abc$,
(*iii*) $4ab \times 5c$,
(*iv*) $5ab \times 4c$, where $\{a, b, c\} = \{1, 2, 3\}$

[$ii > i$] For any product of type (*i*), there is a greater product of type (*ii*), because

$$5 \times abc > 4 \times abc.$$

[$iii > ii$] For any product of type (*ii*), there is a greater product of type (*iii*), because

$$\begin{aligned} 4ab \times 5c &= 4ab \times 50 + 4ab \times c \\ &> 4ab \times 50 + 5 \times c \\ &= 4ab0 \times 5 + c \times 5 \\ &= 4abc \times 5. \end{aligned}$$

[$iii > iv$] For any product of type (*iv*), there is a greater product of type (*iii*), because

$$4ab \times 5c0 > 4a0 \times 5cb, \text{ by the lemma.}$$

So the greatest product must be of type (*iii*). There are three plausible cases of this type:

$$432 \times 51,\ 431 \times 52 \text{ and } 421 \times 53.$$

$$431 \times 52 > 432 \times 51, \text{ by the lemma.}$$

Also

$$431 \times 520 > 421 \times 530, \text{ by the lemma,}$$

so the greatest product is 431×52.