CALCULUS VIA LOGARITHMS?

The usual order in a calculus book is something like this. We begin by learning that differentiating x^n gives nx^{n-1}, where n is a natural number. At some stage it has to be shown that this still works even if n is fractional or negative. (This incidentally involves some slight awkwardness that we had in previous work, in showing that indices could still have a meaning when they were not positive whole numbers.) We learn how to differentiate products and quotients, and we meet the chain rule. Later, some way is found to explain what e^x means and to differentiate it. Finally, ln(x) is defined as the function inverse to e^x.

Now it may well be that generations of teachers have found that this is the order of presentation that suits learners best. However there is a radically different order, in which ln(x) comes in very early, and leads very smoothly and easily to many of the other theorems of calculus. It is certainly worth knowing when we take our second look at calculus.

Imagine we are at the stage where we know how to differentiate powers of x. Treating integration as the reverse of differentiation, we find the integral of x^n is $x^{n+1}/(n+1)$. This is fine for most powers but breaks down if $n = -1$. We are stumped if we are asked to find an area under the curve $y = 1/x$.

There is a regular procedure used by mathematicians when the existing bag of tricks is insufficient to deal with a

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problem that has arisen.. Clearly it is meaningful to speak of the area under $y = 1/x$ between $x=1$ and $x=a$. Very well, let us define a function $L(a)$ by saying that it represents this area. We now study its properties. We hope and expect that this new function will in fact turn out to be the natural logarithm.

The curve $y = 1/x$ has the property that it is unaltered if we multiply all the distances on the x-axis by k and divide those on the y-axis by the same number, k. This transformation leaves areas unaltered. If we find an area by breaking it up into small squares, the transformation changes one of these squares, measuring c by c, into a rectangle ck by c/k, with area unchanged.

When this transformation acts the region under the curve $y = 1/x$ between $x=1$ and $x=a$ changes into the region under the same curve between $x=k$ and $x=ak$. (See Figure 1.) The area under this new region is the difference between the area from $x=1$ to $x=ak$ and the area from $x=1$ to $x=k$. That is to say, its area is $L(ak) - L(k)$. But the new area is the same as the old area. So we have the equation

$$L(a) = L(ak) - L(k).$$

This means $L(ak) = L(a) + L(k)$. Here we have the essential property of logarithms, the logarithm of a product is the sum of the logarithms of the factors.

$L(1)$ of course is zero, as there is no area if we go from 1 to 1. If we put $k=1/a$, and use $L(1)=0$ we find that $L(1/k)= -L(k)$. We have so far considered k as being larger than 1, so that 1/k will be less than 1. Thus L(x) is negative if x lies between 0 and 1. We define L(x) only for positive numbers.

The idea of a negative area might trouble someone, although this idea is continually being used when we integrate some positive function from p to q, where p>q. We could get round this possible difficulty by defining L(a) as the area from h to a **minus** the area from h to 1, where h could be any number conveniently close to zero. The value of L(a) of course would not depend on the choice of h. This extra explanation, incidentally, makes it very clear that L(a) always increases when a increases.

This property is important. In the old days, when multiplication of complicated numbers was done by logarithms, a procedure was followed which gave the logarithm of the answer. A table of antilogarithms was then used, which gave the number having this logarithm. The existence of such a table implied two things, (i) that, for any number, y, you might choose there would be a number, x, such that ln x = y. (ii) that there could not be two different numbers, x, with this property.

The fact that L(a) always grows with a shows that (ii) holds for it. The graph cannot turn down and revisit a height that it was at earlier.

But can we be sure that there will always be a number with a given logarithm? In fact, we can. As we move right, L(a) steadily increases. The only way there could be numbers that were not logarithms of anything, would be if there were a certain minimum, m, that logarithms never went below, or a certain maximum, M, that they never went above. We can in fact prove that neither of these things is so. If either of them had been so, makers of tables of antilogarithms would have been disturbed by it centuries ago.

We can prove L(a) --> +oo as a --> +oo. Figure 2 shows rectangles lying underneath the curve $y = 1/x$. Between $x=1$ and $x=2$, the rectangle has height 1/2, between $x=2$ and $x=3$ it has height 1/3, and for each n from 2 on, the height between $x=n-1$ and $x=n$ is $1/n$. However far we go, the total area of the rectangles, up to a certain value of x, is clearly less than the area under the curve up to that value. So the sum

$$S = 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \ldots\ldots + 1/n$$

gives us an underestimate for L(n), the area between $x=1$ and $x=n$. The terms of this sum are decreasing and at first glance it might seem that the sum will never become very big, but in fact we can make S as large as we like by taking n sufficiently large. For we have

$$1/3 + 1/4 > 1/4 + 1/4 = 1/2.$$

$$1/5 + 1/6 + 1/7 + 1/8 > 1/8 + 1/8 + 1/8 + 1/8 = 1/2.$$

We can continue in this way, grouping terms together. The last term of each group has a power of 2 in the denominator, and the sum of the terms in each group is more than 1/2. By taking enough halves we can make the sum as big as anyone may require.

We know L(a) = -L(1/a). As a --> 0, 1/a --> +oo, so

L(1/a) --> +oo, and L(a) --> -oo.

Thus logarithms fill all the space between -oo and +oo. Every real number is the logarithm of some real number.

L(a) seems to do everything we expect from a logarithm. From now on we shall write it as ln a.

The logarithmic scale.

A special graph paper, used for interpreting scientific experiments, has numbers marked on one or both axes in the way shown in Figure 3. 1 is marked at the origin,O. Each number is marked at the distance given by its logarithm. In Figure 3 we have picked out the powers of 10 and the powers of 2.

You will notice that the powers of 2 are evenly spaced. This is to be expected since ln (2x) = ln x + ln 2. As we multiply by 2 to go from one power of 2 to the next, we know each power will be a distance ln 2 further along the axis than the previous one.

All the powers at one stroke.

We are now in a position to define x^k without bothering whether k is positive, integral, rational or not. Our definition simply is;-

x^k is the number k times as far from the origin as x on the logarithmic scale.

This is a good definition. If we go to the point specified, there will certainly be a number, and only one number, residing at that point. If k is negative, the point for x^k will be on the opposite side of the origin from x. Apart from that, we do not need to bother at all about what

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kind of number k is.

If we look at Figure 3, we can see that 32 is marked at a point 5/4 times as far from the origin as 16. So 32 must be $16^{5/4}$. In the same way, 1/8 is -3/2 times as far from the origin as 4, so $1/8 = 4^{-3/2}$.

In algebraic terms;

Definition. x^k is the number y given by ln y = k ln x.

Finding Laws from Experimental Data.

In log-log graph paper, logarithmic scales are used on both axes. If, for example, data gave points on (or nearly on) a straight line of gradient m through the origin, this would suggest ln y = m ln x, that is to say, $y = x^m$.

Differentiating a power.

We are now in a position to differentiate x^n without making any enquiries about the arithmetical nature of n.

We shall need one calculus principle to do this, the chain rule. If $y = \phi(z)$ and $z = f(x)$ then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \quad .$$

Note that we are entitled to use this principle; it does not depend on any of the theorems that would precede it in the usual order of development.

We need this principle in order to differentiate ln f(x). First of all, we know d/dx ln x = 1/x since ln was defined in terms of an area under the graph of 1/x. Then, as usual when applying the chain rule, we bring in z.

Let $w = \ln z$, $z = f(x)$, so $dw/dz = 1/z$, $dz/dx = f'(x)$.

Hence

$$\frac{dw}{dx} = \frac{dw}{dz}\,\frac{dz}{dx} = \frac{1}{z}\,f'(x) = \frac{f'(x)}{f(x)}$$

So $$d/dx \ln f(x) = f'(x)/f(x). \quad \ldots. \; \text{(I)}$$

We now come to differentiating x^n. The definition of x^n is that it equals y, where

$$\ln y = n \ln x. \quad \ldots \; \text{(II)}$$

Now y here is a function of x, so we may use equation (I) with y in the place of $f(x)$. Accordimngly we find, on differentiating equation (II)

$$y'/y = n/x \quad \text{so } y' = ny/x = nx^n/x = nx^{n-1}.$$

(You may object that we have here used the result $x^n/x = x^{n-1}$ without justification. It is not hard to prove this from the logarithmic definition of powers and the properties of logarithms.)

Differentiating products and quotients.

The formulas for differentiating products and quotients follow very easily in this approach.

Consider the product of 3 functions, $y = uvw$. In terms of logarithms this means

$$\ln y = \ln u + \ln v + \ln w.$$

Differentiate, using equation (I). This gives

$$y'/y = (u'/u) + (v'/v) + (w'/w).$$

Multiplying by uvw gives

$$y' = u'vw + uv'w + uvw'.$$

Clearly we can differentiate in this way the product of any number of factors.

For the quotient $y = p/q$ we have $\ln y = \ln p - \ln q$. Differentiating gives $y'/y = (p'/p) - (q'/q)$. Accordingly

$$y' = (yp'/p) - (yq'/q).$$

As $y = p/q$ this quickly leads to the usual result

$$y' = (p'q - q'p)/q^2.$$

Quite apart from any logical advantages this approach may have, it is quite useful as an aid to remembering these formulas.

The Exponential Function.

There must be some number, the natural log of which is 1. Let us call it e, so $\ln e = 1$. e^x, being a power of e, is defined as the number y for which $\ln y = x \ln e$. As $\ln e = 1$, this means $\ln y = x$. So $y = e^x$ is defined by $\ln y = x$. That is to say, the exponential function is the inverse of the logarithmic function.

Now for $x = \ln y$, we know $dx/dy = 1/y$. This implies $dy/dx = y$. So differentiating e^x gives e^x, as we knew long ago.

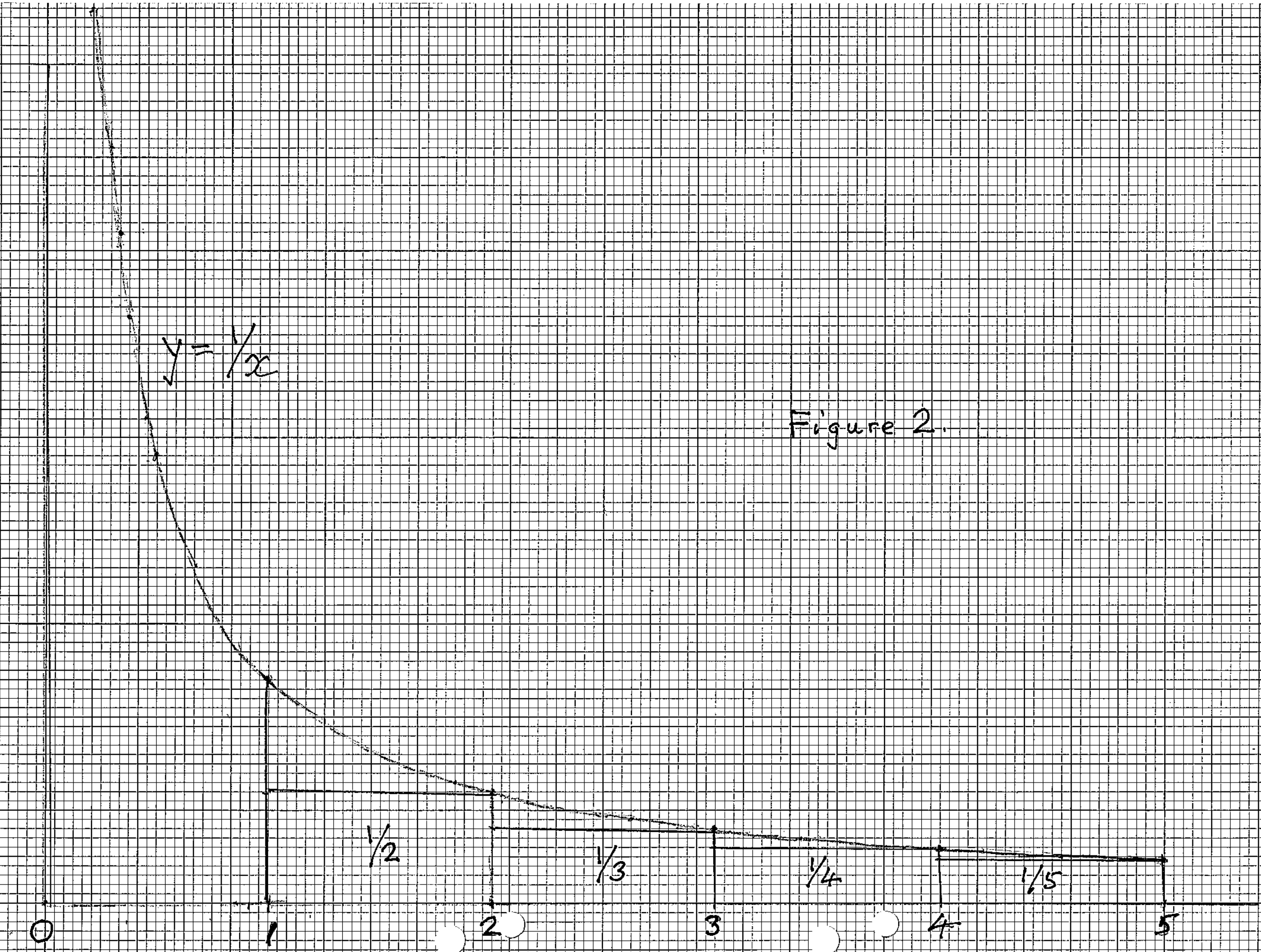

Figure 2.

Figure 2

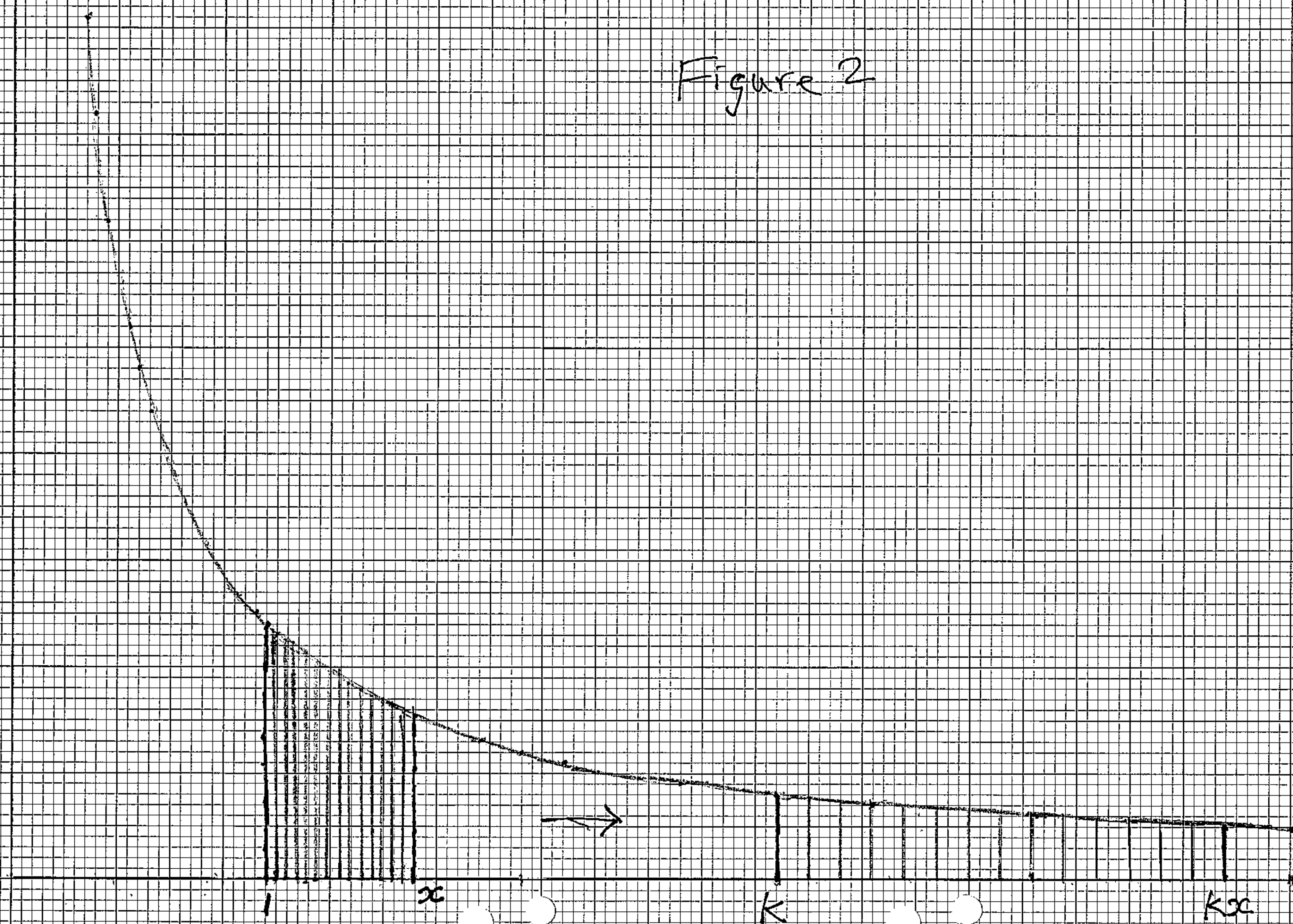

Figure 3. The Logarithmic Scale.