CF 1 C4

Continued fractions.

An expression such as $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$

To express a number as a continued fracta.

Example, $52/23$. $\frac{52}{23} = 2\frac{6}{23}$ $1/\frac{6}{23} = \frac{23}{6}$

$\frac{23}{6} = 3\frac{5}{6}$ $1/\frac{5}{6} = \frac{6}{5}$

$\frac{6}{5} = 1\frac{1}{5}$

$$\therefore \quad \frac{52}{23} = 2 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5}.$$

More interesting, express $\sqrt{2}$ as a continued fraction.

$\sqrt{2} = 1 + (\sqrt{2}-1) = 1 + \frac{1}{\sqrt{2}+1}$

$\sqrt{2}+1 = 2 + (\sqrt{2}-1) = 2 + \frac{1}{\sqrt{2}+1}$

$\sqrt{2}+1 = 2 + (\sqrt{2}-1)$ From now on, we get 2 at each step.

$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\ldots\ldots$ for ever.

It is of course necessary to check that the R.H.S. does converge to a definite limit.

We can break off, as 1, or $1+\frac{1}{2}$, or $1+\frac{1}{2+}\,\frac{1}{2}$ etc.

$1+\frac{1}{2} = \frac{3}{2}$ $1 + \frac{1}{2\frac{1}{2}} = 1\frac{2}{5} = \frac{7}{5}$

Note $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4}$ $\left(\frac{7}{5}\right)^2 = \frac{49}{25} = 2 - \frac{1}{25}$

The sequence continues $\frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$

$\left(\frac{17}{12}\right)^2 = \frac{289}{144} = 2\frac{1}{144}$; $\left(\frac{41}{29}\right)^2 = \frac{1681}{841} = 2 - \frac{1}{841}$

$\left(\frac{99}{70}\right)^2 = \frac{9801}{4900} = 2 + \frac{1}{4900}$

It seems that we get ever better approximations, alternately above and below.

Convergents.

If we break off $a_1 + \frac{1}{a_2+} \frac{1}{a_3+} \frac{1}{a_4+} \cdots$ at various points we get

$$\frac{p_1}{q_1} = \frac{a_1}{1}; \quad \frac{p_2}{q_2} = \frac{a_1 a_2 + 1}{a_2}; \quad \frac{p_3}{q_3} = \frac{a_1 a_2 a_3 + a_1 + a_3}{a_2 a_3 + 1}$$

We observe

$$p_3 = a_3 p_2 + p_1$$
$$q_3 = a_3 q_2 + q_1 .$$

This suggests that perhaps

$$p_n = a_n p_{n-1} + p_{n-2} \qquad (\mathrm{Ia})$$
$$q_n = a_n q_{n-1} + q_{n-2} \qquad (\mathrm{Ib})$$

<u>Proof by induction</u> This holds for $n = 3$. If it holds as far as $n = N$. p_{N+1}/q_{N+1} is formed by going as far as a_{N+1}. This means that we replace

$$a_1 + 1/a_2 + \cdots + 1/a_N$$

by

$$a_1 + \frac{1}{a_2} \quad + \frac{1}{a_N + \frac{1}{a_{N+1}}}$$

So we get p_{N+1}/q_{N+1} by changing a_N to $a_N + \frac{1}{a_{N+1}}$ in the equations for p_N/q_N.

i.e.

$$\frac{p_{N+1}}{q_{N+1}} = \frac{(a_N + \frac{1}{a_{N+1}})p_{N-1} + p_{N-2}}{(a_N + \frac{1}{a_{N+1}})q_{N-1} + q_{N-2}}$$

$$= \frac{a_N a_{N+1} p_{N-1} + p_{N-1} + a_{N+1} p_{N-2}}{a_N a_{N+1} q_{N-1} + q_{N-1} + a_{N+1} q_{N-2}}$$

$$= \frac{a_{N+1}(a_N p_{N-1} + p_{N-2}) + p_{N-1}}{a_{N+1}(a_N q_{N-1} + q_{N-2}) + q_{N-1}}$$

$$= \frac{a_{N+1} p_N + p_{N-1}}{a_{N+1} q_N + q_{N-1}}$$

Thus, if the formula is correct up to $n = N$ it is correct up to $n = N+1$.

The convergents (i.e p_n/q_n) to $\sqrt{2}$ are

$$\frac{1}{1} \quad \frac{3}{2} \quad \frac{7}{5} \quad \frac{17}{12} \quad \frac{41}{29} \quad \frac{99}{70} \quad \ldots$$

$$-\frac{1}{1}+\frac{3}{2} = +\frac{1}{2} : -\frac{3}{2}+\frac{7}{5} = \frac{-1}{10} : -\frac{7}{5}+\frac{17}{12} = \frac{+1}{60} :$$

$$-\frac{17}{12}+\frac{41}{29} = \frac{-1}{12\times 29} \qquad -\frac{41}{29}+\frac{99}{70} = \frac{+1}{29\times 70}$$

These results suggest (1) the numerator is always ± 1, (2) the denominator is the product of the denominators.
i.e it is suggested $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(-1)^n}{q_n q_{n-1}}$. (IIa)

~~Proof by induction Suppose this true as far as~~

This means $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$. (IIb)

Proof by Induction If this is true as far as $n = N$
the next expression is $p_{N+1} q_N - q_{N+1} p_N$

$$= (a_{N+1} p_N + p_{N-1}) q_N - (a_{N+1} q_N + q_{N-1}) p_N$$
$$= p_{N-1} q_N - p_N q_{N-1} = -(-1)^N = (-1)^{N+1}.$$

Corollary. The convergents, as found by the rule, are in their lowest terms (III)
For if p_n and q_n had a common factor, it would be a factor of $(-1)^n$.

The equation $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(-1)^n}{q_n q_{n-1}}$ shows that,
if n is even, $\dfrac{p_n}{q_n} > \dfrac{p_{n-1}}{q_{n-1}}$; if n is odd (IV)
$\dfrac{p_n}{q_n} < \dfrac{p_{n-1}}{q_{n-1}}$.

Corollary. If p, q are two integers with no common factor, we can find integers p', q' :- $pq' - p'q = 1$.
Proof. Express p/q as a continued fraction. Let p'/q' be the convergent before the final p/q.

It follows from Ia and Ib that $p_n > p_{n-1}$ and $q_n > q_{n-1}$

(V)

(VI) $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} a_n.$

Proof From I

$$p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2})$$
$$= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = a_n (-1)^{n-1}.$$

(VII) It follows from (VI) that

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1} a_n}{q_n q_{n-2}}.$$

(VIII) Hence the even convergents continually decrease, while the odd convergents continually increase.

(IX) Hence even p_n/q_n decrease to a limit, the odd p_n/q_n increase to a limit.

(X) These limits are the same, for

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$$ and RHS $\to 0$ by (V).