COVARIANT AND CONTRAVARIANT VECTORS.

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Some time ago I received a letter from an American correspondent. He is not a scientist but has been making a serious effort to understand science. He sent me various offprints giving explanations of covariant and contravariant, none of which made sense to him.

The root of the difficulty seems to be terminology. If a vector is defined as an ordered set of numbers, (5,6,7,8) is a vector and there is no conceivable way of classifying it as covariant or contravariant. For tensor theory or for applications to relativity it seems better to say that a vector is a geometrical or physical entity, that can be specified by an ordered set of numbers, and that this specification can be done either in the covariant or contravariant manner. This can be illustrated very simply in 2 dimensions and without using differentials.

We suppose the axes of co-ordinates are at an angle A, which is not a right-angle. A displacement D, given by the vector OP, will naturally be represented by the co-ordinates of P, taken in the directions of the axes. Thus D in Figure 1 is represented by (x^1, x^2). The work done by the force F in the displacement OP is $x^1F_1+x^2F_2$ where F_1 and F_2 are the components of the force in the directions of the axes.

The method we have used for specifying the force is called covariant, and that for the displacement contravariant.

The work done by a force in a displacement is given by the scalar product, so the expression found above gives a formula for a scalar product. In it, one vector appears in its contravariant , the other in its covariant form.

It may happen that we need to calculate the scalar product of two vectors both of which have appeared only in contravariant form. The covariant form is easily deduced from the contravariant. The components of OP in the directions of the axes are $x^1 + x^2 \cos A$ and $x^1 \cos A + x^2$. These give us x_1 and x_2 and we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & \cos A \\ \cos A & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

If we denote the elements of the matrix here as g_{ij} we have $x_i = \sum g_{ij}\, x^j$.

To find the length of OP we take the scalar product of D with itself, that is, we need

$$\sum x^i x_i = \sum\sum g_{ij} x^i x^j.$$

Thus the tensor used for lowering or raising indices is necessarily the same as that involved in the formula for distances.

<u>Note.</u> At this stage it is not convenient to use the convention by which lower subscripts are used for co-ordinates, instead of the upper position that is logically natural.

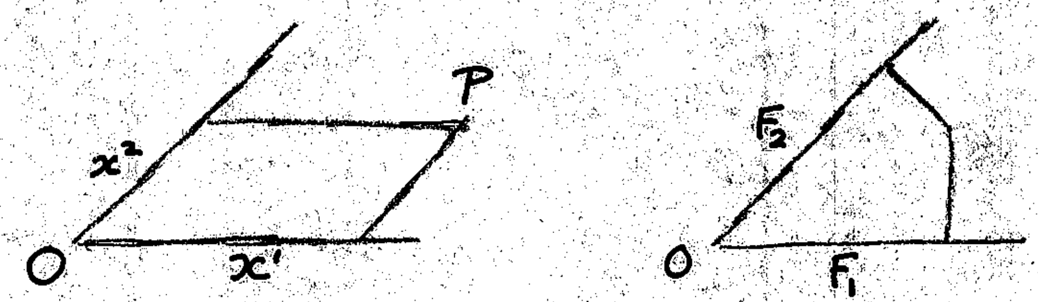

Figure 1.