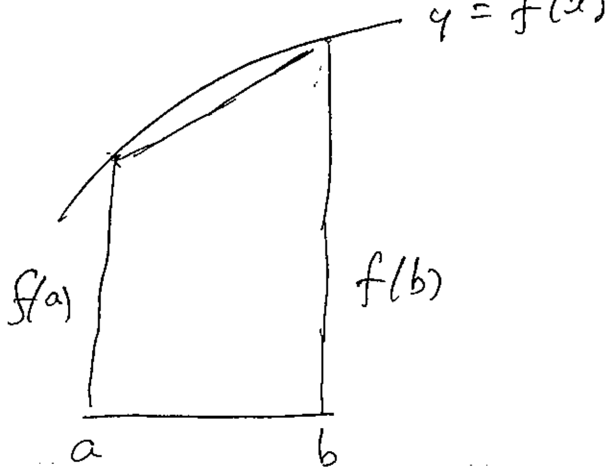
ERROR IN $\int$ BY TRAPEZOID RULE   TRAPEZOID ERROR

Ref: section I J in Sat am BINDER

I 1

A straight line $y = mx + c$ is to join the points $(a, f(a))$, $(b, f(b))$. So



$$f(a) = ma + c$$
$$f(b) = mb + c$$
$$\therefore\ m = \frac{f(b) - f(a)}{b - a}$$
$$c = \frac{b f(a) - a f(b)}{b - a}$$

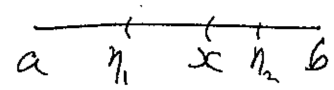
So $$y = \frac{f(b) - f(a)}{b - a} x + \frac{b f(a) - a f(b)}{b - a}$$
$$= \frac{1}{b-a} \{ f(a)(b - x) + f(b)(x - a) \} = p(x).$$

$y = p(x)$ is the line joining A to B, and it gives an approximation to $f(x)$. It is found convenient to express the error in the form $f(x) - p(x) = R(x)(x-a)(x-b)$. (1)

This is reasonable, as clearly the error is 0 for $x = a$ and for $x = b$. An expression for the error is found as follows.

Let $G(z) = f(z) - p(z) - R(x)(z - a)(z - b)$

Eq. (1) shows that $G(x) = 0$ for any x. In particular $G(z) = 0$ for a, b and for <u>any</u> x between them.



As $G(z) = 0$ for $z = a$ and $z = x$, $G'(z) = 0$ for $z = \eta_1$, $a < \eta_1 < x$.

As $G(z) = 0$ for $z = x$ and $z = b$, $G'(z) = 0$ for $z = \eta_2$, $x < \eta_2 < b$

As $G'(z) = 0$ for $z = \eta_1$ and for $z = \eta_2$, we must have $G''(\xi) = 0$ for some ξ between η_1 and η_2.

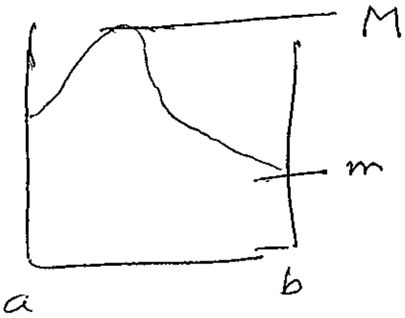
$$G''(z) = f''(z) - p''(z) - 2R(x) \qquad p''(z) = 0.$$
$$\therefore\ 0 = G''(\xi) = f''(\xi) - 2R(x).$$
$$\therefore\ R(x) = \tfrac{1}{2} f''(\xi) \text{ for some } \xi \text{ in } (a, b).$$

The value of ξ will depend on the value of x.

If we know that $f''(x)$ lies between $-L$ and $+M$ in (a, b), we can be sure the error at x will lie between $-\frac{1}{2}L(x-a)(x-b)$ and $\frac{1}{2}M(x-a)(x-b)$.

The generalized mean-value theorem

The particular mean-value theorem is, if $f(x)$ is continuous in $[a,b]$ then $\int_a^b f(x)\,dx = f(\xi)(b-a)$ for some ξ in $[a,b]$



Proof Let M be max $f(x)$ and $m = \min f(x)$ in (a,b).

Clearly

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Since $f(x)$ is continuous, it takes every value between m and M. Hence, for some ξ $f(\xi) = \dfrac{\int_a^b f(x)\,dx}{b-a}$

Generalized Theorem.

$f(x)$ continuous in $[a,b]$, $\varphi(x) > 0$ in $[a,b]$.

Then $\int_a^b f(x)\varphi(x)\,dx = f(\xi)\int_a^b \varphi(x)\,dx$. ξ in $[a,b]$

Proof It is convenient, though not necessary, to suppose $\varphi(x)$ continuous. Let $\psi(x) = \int^x \varphi(t)\,dt$.

Then $d\psi(x) = \varphi(x)\,dx$.

$$\int_a^b f(x)\,\varphi(x)\,dx = \int_{\psi(a)}^{\psi(b)} f(x)\,d\psi(x)$$

$$= f(\xi)\,[\psi(b) - \psi(a)]$$

$$= f(\xi)\int_a^b \varphi(x)\,dx. \qquad \text{Q.E.D.}$$