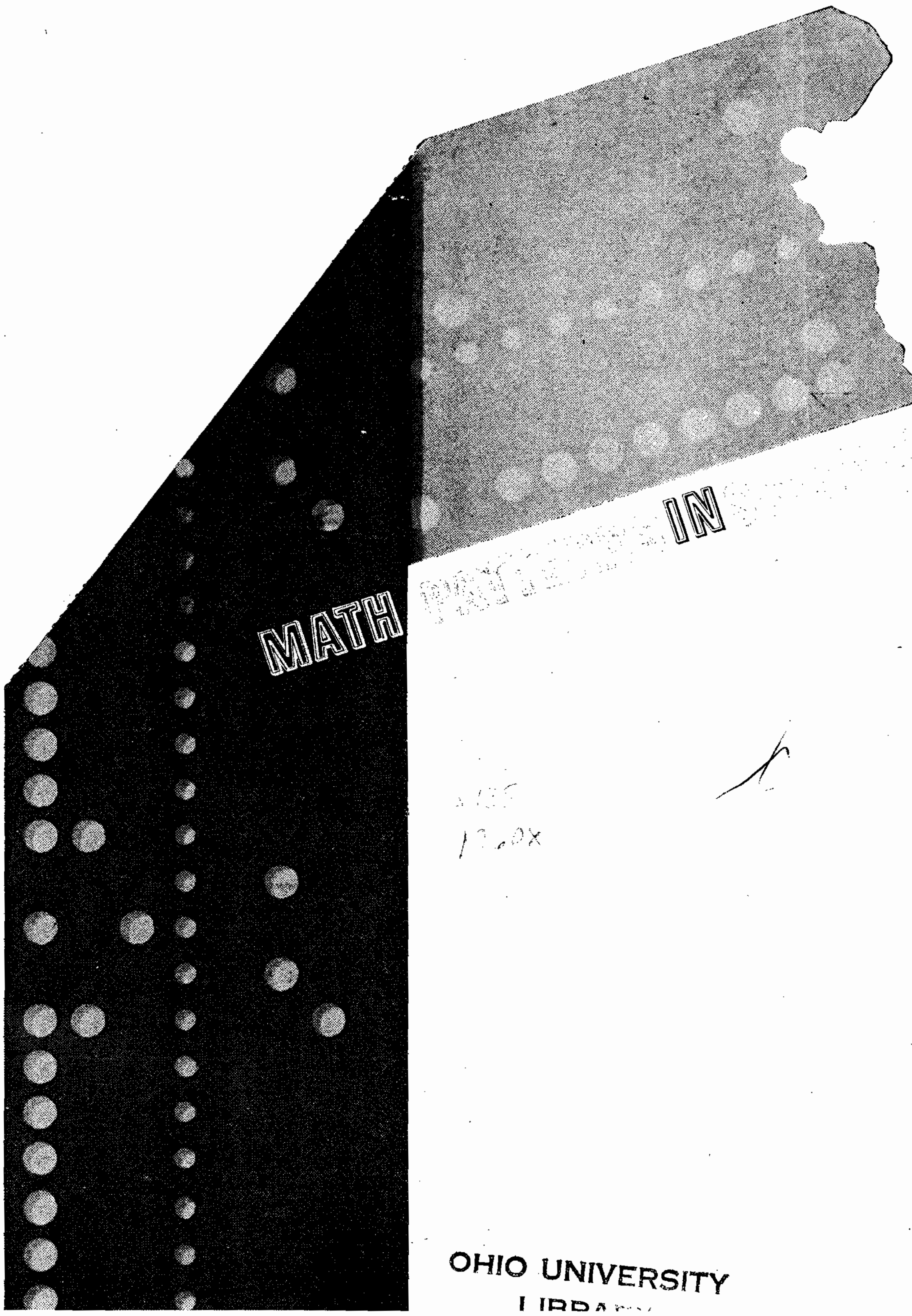

MATH [illegible] IN [illegible]

BY W. W. SAWYER

INTRODUCTION

THIS book tries to show why nearly all mathematicians and scientists find their work beautiful. They see patterns in nature and in number. In this book, students will have an opportunity to discover some of these patterns for themselves.

No previous knowledge of science or mathematics is required, as this book can be read by anyone who knows the basic facts of arithmetic. A student can use it for individual study. It can, of course, be used for class study.

Answers to most of the questions are given on page 31. The student should check his own answers with those on that page, and he should do this immediately after deciding upon his own result. He will in this way discover at once if he has understood what the author has been trying to communicate.

Most students will find that they can think their way successfully through this book. The work, however, is not spoon-feeding, and the student is given every opportunity to use his own judgment—to collect evidence, to make guesses, to observe, to invent.

In some cases a scientific result may be given without going into detail about its background. For example, we shall learn that a stone falls 64 feet in 2 seconds. We shall have to accept for the moment the fact that competent scientists have agreed upon this, even though we cannot prove the statement without complicated apparatus. It is not unscientific to accept the results of other people's experiments.

But such quoting of evidence lies on the fringes of this book. Its main purpose is to assemble material in which the student can see clearly what is happening and from which he can draw his own conclusions.

This is something which, as a rule, students enjoy doing.

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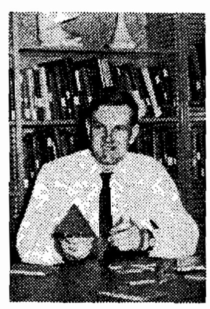
THE AUTHOR

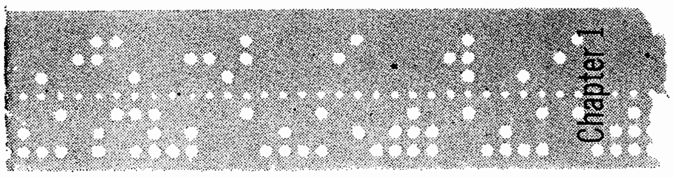


Mr. Sawyer is now Professor of Mathematics in Wesleyan University, Middletown, Connecticut. The unusual story of his life takes him from his own university (Cambridge, in England) through teaching experience in England, in Ghana, in New Zealand, and now in the United States. In those years he has taught a wide range of mathematics, all the way from the elementary school through the university graduate school.

Professor Sawyer believes deeply that students should become aware of the relationship between science and mathematics long before they reach the college. Since joining the Wesleyan faculty he has demonstrated this interest by organizing and conducting a math club for students in the Middletown (Conn.) public schools. This voluntary group had a marked success.

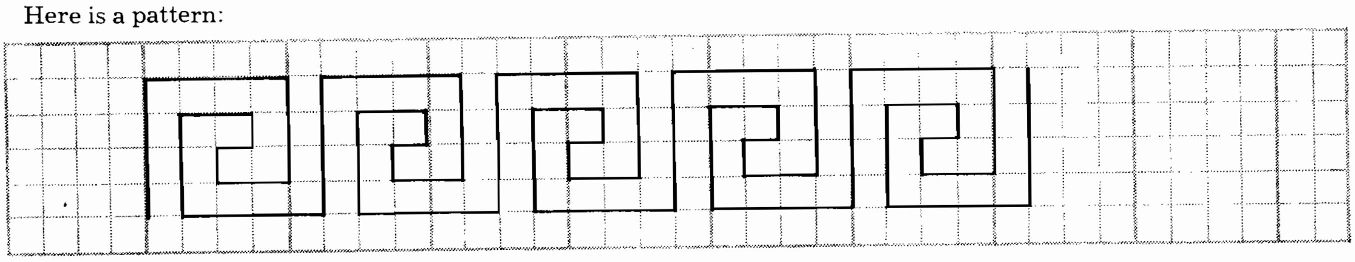
Our author is the editor of "The Mathematics Student Journal," a quarterly published by the National Council of Teachers of Mathematics. He is an author and lecturer of note.

AMERICAN EDUCATION PUBLICATIONS Education Center, Columbus 16, Ohio

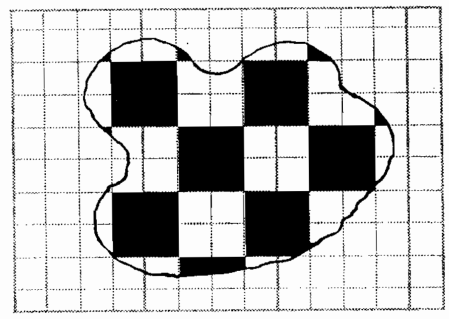


RECOGNIZING PATTERNS

Here is a pattern:



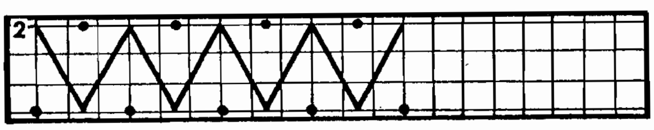
A pattern is often interesting to look at. Also, it tells you something. This pattern leaves off before it reaches the edge of the page. But you can guess how it would go on to fill the line on the page.

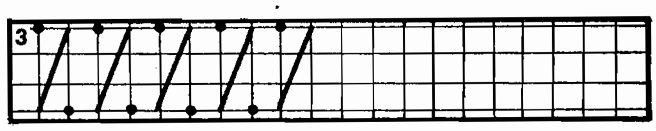


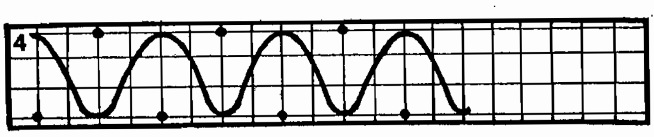
Patterns To Complete

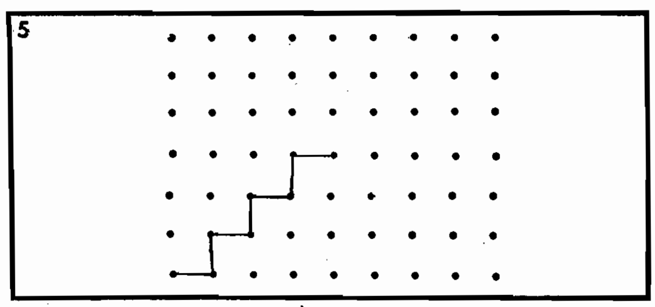
1. A man is digging and he discovers an ancient pavement. He can see only the part in the middle, because soil covers the rest of it. However, by looking at the pattern in the part that he sees, he can guess what the rest will look like. Shade in what you think he will find under the soil.

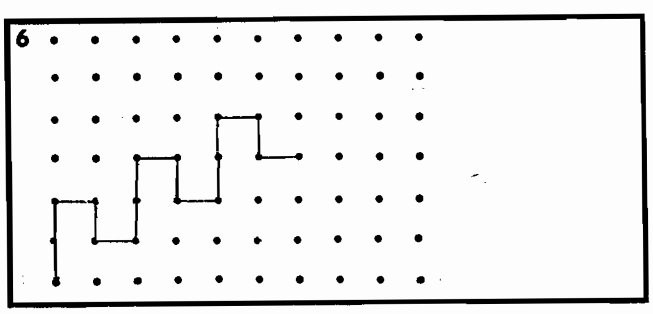
In each pattern below, the right hand part is incomplete. *Draw in each what you think should be there.*

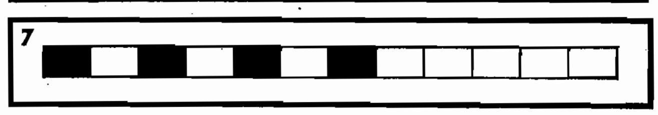


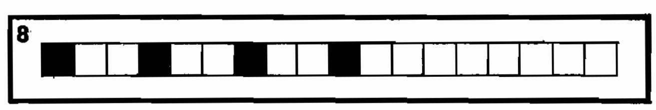


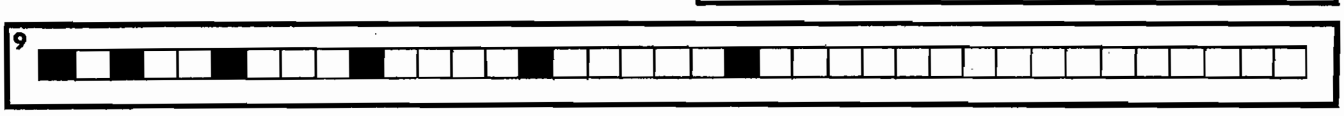


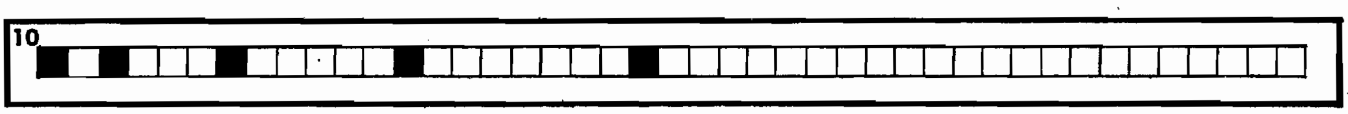


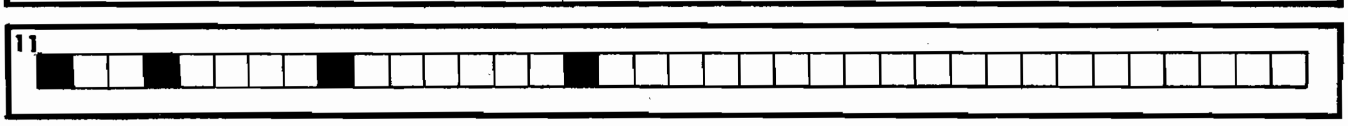












PATTERNS IN NATURE

Patterns are of real value in science, and indeed in all of life. Science comes from the Latin word for "knowledge." Whenever we know that something will happen in the future—perhaps it would be better to say whenever we expect something to happen in the future—it is because we have seen a pattern in past events.

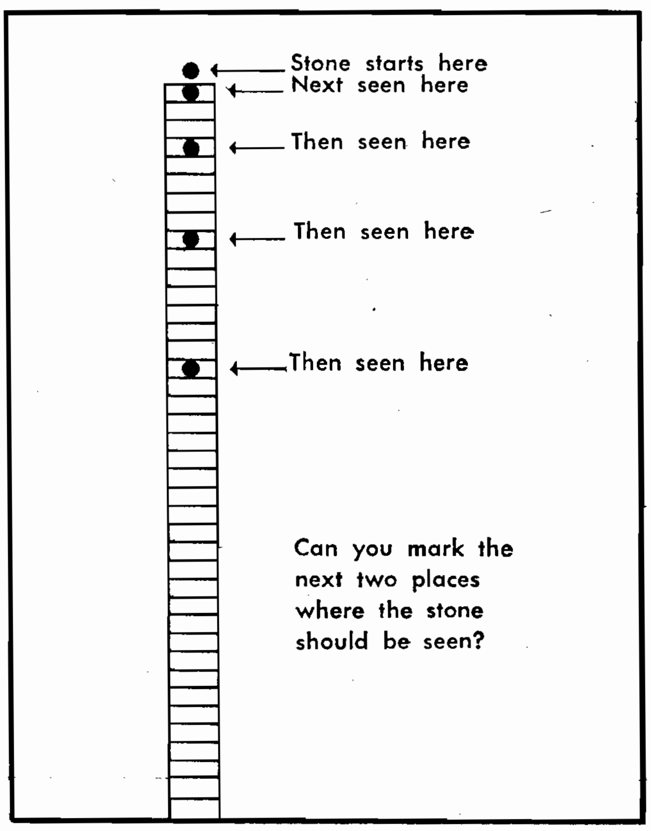
There is the pattern of the seasons—winter, spring, summer, autumn. There is the pattern of human life—birth, childhood, youth, parenthood, old age, death. We should be very surprised if a baby grew into an old man and then became a boy.

There is a certain order of events that we are used to, and we expect this order to continue.

Patterns do not always exist without change. In some exceptional years people say, "Really we had no spring at all this year." Many forces act to give us our weather, and it is particularly difficult to find a pattern for forecasting the weather.

In the same way, it is usually impossible to find a pattern in the way a feather, a leaf, or a light piece of paper falls. The air acts on a falling leaf and causes it to perform a beautiful but complicated dance. No pattern can be seen. If, however, we drop a stone, its motion is much simpler and we can detect a simple pattern in it.

To discover this pattern, imagine a stone falling in utter darkness. And imagine that we have a machine that will flash a light every quarter-second. Each time the light flashes, we can see where the stone is and we can take a picture of it. We should obtain a result something like the diagram below.



PATTERNS IN NUMBERS

There are several different ways of answering the question about the falling stone, but, however you do it, you need to count the squares through which the stone has fallen. So you are looking for a *pattern in numbers.*

Numbers come into most branches of science and engineering. How fast is the earth traveling? How far is it from the earth to the moon? How thick is a bar of steel? What weight can it safely support? What is the proportion of salt in sea water? To what temperature must water be raised to kill the germs in it?

All these questions, and many others, have something to do with numbers. To discover the laws of science, you need to be good at spotting patterns in numbers, and much of this booklet will deal with how to do that.

You may not have realized how much patterns come into arithmetic. The exercises below show that there are patterns even in the multiplication tables.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99
100	101	102	103	104	105	106	107	108	109

Squares have been drawn around 9, 18, 27, because these numbers are in the 9-times table. **Mark in the same way all the numbers of the 9-times table;** that is, all the numbers that are exactly divisible by 9.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49

Mark all the numbers that are exactly divisible by 2; that is the even numbers.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69

Mark all the numbers that are in the 3-times table.
(Do not stop at "three tens or three twelves." As 69 = 3 × 23, 69 counts as being in the 3-times table.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69

Mark all the numbers that are in the 4-times table.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Mark all the numbers that are in the 6-times table.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Mark all the numbers that are in the 7-times table.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Mark all the numbers that are in the 8-times table.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

Mark all the numbers that are in the 10-times table.

Something To Make

You may like to make "frames" for the multiplication tables.

Write the numbers 0 through 99 on graph paper, as in diagram A to the right. We call this the "number sheet."

Then take another piece of graph paper of exactly the same size. In this sheet, cut away some of the squares to leave holes. Suppose, as an example, you cut out squares as indicated on diagram B on this page. We call the paper with the holes cut in it a "frame."

Now put the frame over the number sheet. You will see something like diagram C.

The numbers which show through are numbers of a certain multiplication table. (Which one?)

In the same way, you can make a frame that allows the numbers of the 4-times table to show through. You can make another for the 6-times table. In fact you can make one for each multiplication table.

A Suggestion

In making the frames, you have to be careful with tables like 2-times, 3-times, and 9-times. Each of these frames is in danger of breaking into several pieces.

You may prevent this by leaving bars between the squares to hold the frame together. The number sheet would have to look like D below.

The frame, with the numbers showing through for part of the 2-times frame, would look like E below.

Make the number sheet first, since the same number sheet is used with all the different frames. Then make the frames so that the numbers you want will show through. The black parts of the number sheets will come under the bars which hold the frame together.

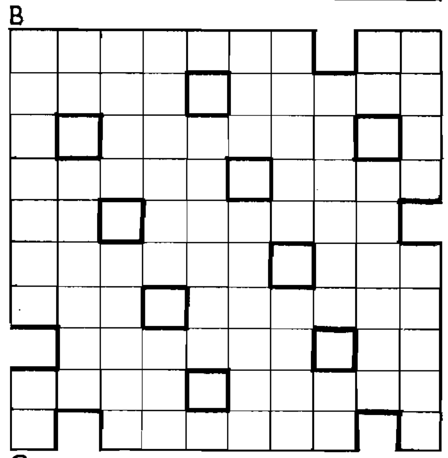
If you do not want to have bars, as suggested above, you may be able to find some other way of holding the frames together. For instance, you might begin with a sheet of transparent paper or cellophane, and glue pieces of paper to it so as to cover the numbers you want to hide.

You can make many experiments with these frames. You can look at a frame and try to guess which table it shows before you put it on the number sheet. You can find out what happens if you put two frames over the number sheet at the same time. Do you expect to see any numbers at all if you put the 2-times and the 3-times frames on? If so, which numbers would show? What do you see if you put the 6-times and the 8-times frames on together?

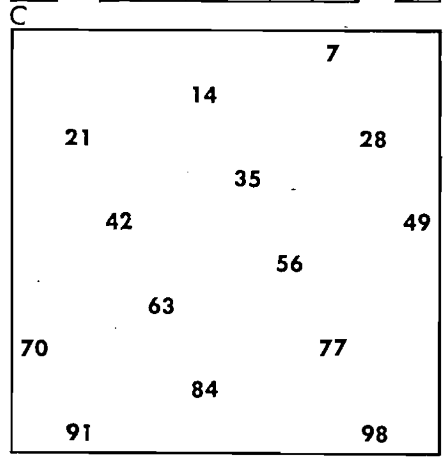
A

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
20	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**
30	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**
50	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**
60	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**
70	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**
80	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**
90	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**

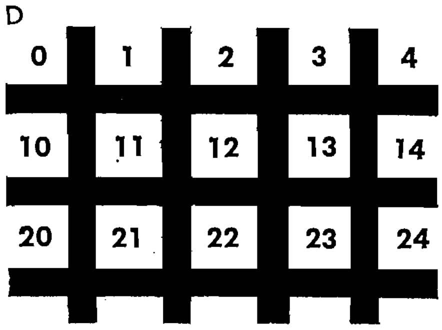
B



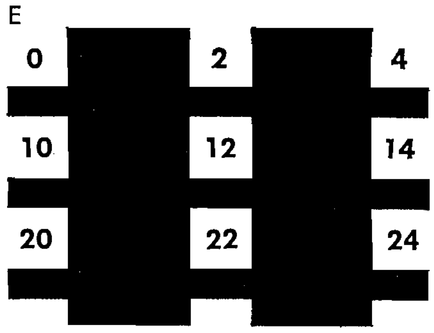
C



D



E



A Guessing Game

This is an interesting game. It is fun and it is useful, for it trains you to make mathematical and scientific discoveries.

One student stands up. One at a time, the other students call out numbers. The student who is standing answers with numbers, but he must first decide what rule he will use. He decides the number he will call by doing the same thing to each number given by a member of the class. He may decide that he will add **2** to each number, or that he will multiply each by **3**. That is what we mean by deciding the rule he is to use.

The rest of the class is to guess from his answers what rule he is using. For example:

1. Henry stands up. He thinks, "Whatever number they say, I will answer one more." Someone calls out **8**, and Henry answers **9**. Someone calls **3**, and Henry answers **4**.

Now Jane thinks that she has guessed what Henry is doing. She does not blurt out the rule, but when someone says the next number, perhaps **15**, Jane says, "Henry I think you are going to answer **16**." When enough students have guessed what Henry is doing, someone tells the actual rule and Henry sits down.

2. Ann stands up. She decides that, whatever number is said, she will double it.

Bill: **7.**

Ann: **14.**

Sue: **3.**

Jack: I think you are going to say **10**. Jack thinks Ann is adding **7** to each number called.

Ann: No, I am going to answer **6**.

Fred: **1**.

Joe: I think you are going to answer **2**.

Ann: You are right.

3. Joe stands up. We have to guess what he is doing.

Susie says **5**.

Joe answers **995**.

Jack says **7**.

Jane: I think you are going to say **997**.

Joe: No. I say **993**.

Ann: **123**.

Joe: **877**.

Susie: **911**.

Joe: **89**.

Alf: A million.

Joe: Say something smaller.

Alf: All right. **10,000**.

Joe: Still too big.

Alf: Well, **1,000** then.

Joe: Zero.

Jane: **999**.

Joe: **1**.

Jane: **998**.

Joe: **2**.

Fred: **600**.

Jane: I think Joe will answer **400**.

Joe: That's right.

What was Joe doing? What rule was he using? ....

....

....

4. Jane stands up. It will save space if we show in two columns the numbers the class calls out, and the answers Jane gives.

Someone calls:	Jane answers:
10	5
42	21
13	$6\frac{1}{2}$
9	$4\frac{1}{2}$
$2\frac{1}{2}$	$1\frac{1}{4}$
6	3
8	4
100	50
20	....
14	....
1	....

Can you fill in the three missing answers? What rule was Jane using? Choose one of these sentences and complete it. Jane is:

Adding .... to each number.

Subtracting .... from each number.

Multiplying each number by ....

Dividing each number by ....

5. Sue stands.

Someone calls:	Sue answers:
7	9
10	12
1	3
4	6
$\frac{1}{2}$	$2\frac{1}{2}$
11	13

Sue's rule is ....

....

....

6.

Someone calls:	Fred answers:
10	7
19	16
5	2
9	6
100	97
82	79
11	8

Fred's rule is ....

....

....

7.

Someone calls:	Bill answers:
1	9
9	1
7	3
5	5
2	8
6	4
12	Too big
11	Too big
8	2

Bill's rule is ....

....

....

8.

Someone calls:	Cathy answers:
6	18
10	30
5	15
100	300
21	63
2	6
3	9
4	12
17	51

Cathy's rule is ....

....

....

9.

Someone calls:	Jack answers:
18	$4\frac{1}{2}$
8	2
100	25
101	$25\frac{1}{4}$
12	3
11	$2\frac{3}{4}$
40	10
44	11
36	9

Jack's rule is ....

....

....

10.

Someone calls:	Nancy answers:
2	4
5	25
3	9
1	1
4	16
10	100
100	10,000
6	36
$\frac{1}{2}$	$\frac{1}{4}$
7	49
1,000	1,000,000

Nancy's rule is different from all the rules we have had so far. Can you find out what she is doing?

If someone calls out	Nancy will answer
8	....
9	....
12	....
11	....

Nancy's rule is ....

....

....

Chapter 2

PATTERNS IN ARITHMETIC

We are now going to hunt for patterns in arithmetic.

To get started, we will do together one question which we shall call QUESTION A1. **Work out 0×0, 1×1, 2×2, 3×3, 4×4, and so on. What do you notice about the answers?**

A1

The answers are:

$0 \times 0 = 0$
$1 \times 1 = 1$
$2 \times 2 = 4$
$3 \times 3 = 9$
$4 \times 4 = 16$
$5 \times 5 = 25$
$6 \times 6 = 36$

Of course, we could go on with this as long as we liked. Our question is, "What do you notice about the answers?"

Perhaps we notice several things. Someone might see that 0 is even, 1 is odd, 4 is even, 9 is odd, and so on. We get even and odd numbers by turns. That is quite true. But it is to be hoped that you would notice the way in which these numbers rise, the change from each number to the next.

From 0 to 1 is an increase of 1
From 1 to 4 is an increase of 3
From 4 to 9 is an increase of 5
From 9 to 16 7
From 16 to 25 9
From 25 to 36 11

You will notice here that we have the odd numbers 1, 3, 5, 7, 9, 11, in order. (What would you expect the next increase to be? Continue with 7×7 and see if you are right.) You will notice that these odd numbers rise each time by 2.

We might write the whole thing like this:

Our answers	0	1	4	9	16	25	36
Rise by		1	3	5	7	9	11
Which rise by			2	2	2	2	2

Here are some more questions of the same kind for you to work out.

A2

0×1
1×2
2×3
3×4
4×5
5×6

Our answers are
They rise by
And these rise by

A3

$0 \times 2 =$ ....
$1 \times 3 =$ ....
$2 \times 4 =$ ....
$3 \times 5 =$ ....
$4 \times 6 =$ ....
$5 \times 7 =$ ....

Our answers are
They rise by
and these rise by

A4

$0 \times 3 =$ ....
$1 \times 4 =$ ....
$2 \times 5 =$ ....
$3 \times 6 =$ ....
$4 \times 7 =$ ....
.... = ...

(Fill in this last line so as to continue the pattern.)

Our answers are
They rise by
and these rise by

A5

$0 \times 4 =$ ....
$1 \times 5 =$ ....
$2 \times 6 =$ ....
$3 \times 7 =$ ....
.... = ...
.... = ...

Our answers are
They rise by
and these rise by

A6

Make up some more questions of this kind for yourself. What do you expect to notice about the answers?

SURPRISING ANSWERS

In the next lot of questions there will also be something new to notice about the answers.

Before we get on to that, we must make sure we understand the same thing by the question. For instance, if I ask, "What is $7 - 3 \times 2$?" I expect 3×2 is 6 and $7 - 6$ is 1. You might say, "But couldn't it be 8? For $7 - 3$ is 4 and 4×2 is 8." Well, it all depends what the question is. 1 is the correct answer to "Multiply 3 by 2, and subtract the answer from 7." 8 is the correct answer to, "Subtract 3 from 7 and then multiply by 2."

Mathematicians have agreed among themselves, to avoid this kind of misunderstanding, that in an expression like $7 - 3 \times 2$, the multiplication is to be carried out before the subtraction. In the old arithmetic books this used to be emphasized very strongly, but not all the new books mention it.

In section B, all our questions have multiplication signs and subtraction signs. The multiplications are to be done before the subtractions. Thus, in question B1, for example, $3 \times 3 - 2 \times 4$ means $9 - 8$, which is 1. You may only subtract after you have done both the multiplications. You will not get the little surprise if you read the question some other way.

B1

$2 \times 2 - 1 \times 3 =$ ....

$3 \times 3 - 2 \times 4 =$ ....

$4 \times 4 - 3 \times 5 =$ ....

$5 \times 5 - 4 \times 6 =$ ....

$6 \times 6 - 5 \times 7 =$ ....

What do you notice? ....

What do you guess $17 \times 17 - 16 \times 18$ will be? .... Work it out and see if it is. What do you think $13{,}589 \times 13{,}589 - 13{,}588 \times 13{,}590$ will be? ....

B2

$2 \times 3 - 1 \times 4 =$ ....

$3 \times 4 - 2 \times 5 =$ ....

$4 \times 5 - 3 \times 6 =$ ....

$5 \times 6 - 4 \times 7 =$ ....

What do you notice? ....

What do you guess $86 \times 87 - 85 \times 88$ is? .... Guess $101 \times 102 - 100 \times 103$ and check by working it out.

B3

$2 \times 4 - 1 \times 5 =$ ....

$3 \times 5 - 2 \times 6 =$ ....

$4 \times 6 - 3 \times 7 =$ ....

$5 \times 7 -$ .. $\times$ .. $=$ ....

$6 \times$ .. $-$ .. $\times$ .. $=$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

What do you notice? ....

Use the result to guess the answer to a question with large numbers in it. ....

B4. Work out

$3 \times 6 - 1 \times 8 =$ ....

$4 \times 7 - 2 \times 9 =$ ....

$5 \times 8 - 3 \times 10 =$ ....

and use this to make some guesses. ....

B5. Work out

$2 \times 6 - 3 \times 4 =$ ....

$3 \times 7 - 4 \times 5 =$ ....

$4 \times 8 - 5 \times 6 =$ ....

$5 \times 9 - 6 \times 7 =$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

What do you notice about the answers? Is this question like questions B1 through B4 or is something different happening now? ....

B6. Work out

$3 \times 8 - 4 \times 6 =$ ....

$4 \times 9 - 5 \times 7 =$ ....

$5 \times 10 - 6 \times 8 =$ ....

$6 \times 11 - 7 \times 9 =$ ....

.. $\times$ .... $-$ .. $\times$ .. $=$ ....

B7. Work out

$3 \times 4 - 1 \times 5 =$ ....

$4 \times 5 - 2 \times 6 =$ ....

$5 \times 6 - 3 \times 7 =$ ....

$6 \times 7 - 4 \times 8 =$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

B8. Work out

$1 \times 3 - 0 \times 2 =$ ....

$2 \times 4 - 1 \times 3 =$ ....

$3 \times 5 - 2 \times 4 =$ ....

$4 \times 6 - 3 \times 5 =$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

Anything new here? ....

B9. Work out

$1 \times 1 - 0 \times 0 =$ ....

$2 \times 2 - 1 \times 1 =$ ....

$3 \times 3 - 2 \times 2 =$ ....

$4 \times 4 - 3 \times 3 =$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

.. $\times$ .. $-$ .. $\times$ .. $=$ ....

B10. Something happened in questions B1 through B4 that was different from what happened in questions B5 through B9. Make up for yourself many questions like the ones above. Sort out those that behave like B1 through B4. Can you find any way of making up questions that will be sure to be like questions B1 through B4? Can you find a way of getting questions that will behave like B5 through B7? Can you make up questions that will behave like B8 and B9?

TRICKS AND WHY THEY WORK

Chapter 3

In Chapter 2 we had some questions with surprising answers. In this chapter we look at another kind of surprise—tricks with numbers.

Probably you have often heard tricks like this one:

> **Think of a number.**
> **Add three to it.**
> **Double.**
> **Take away four.**
> **Halve.**
> **Take away the number you first thought of.**

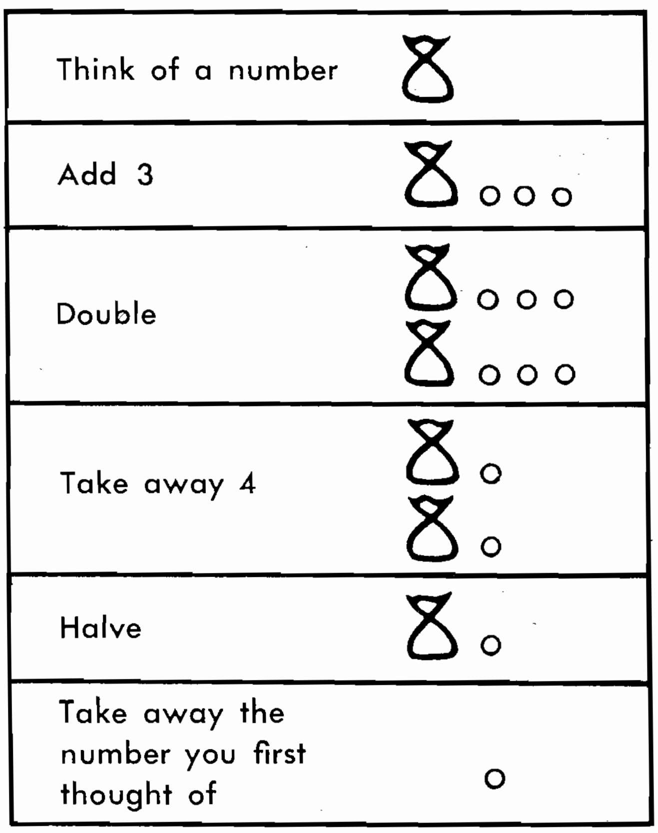
Whatever number you think of at the start of the trick, provided you make no mistakes, you will end up with the answer **1**.

Why does this work? We can see by thinking in pictures. When you have thought of your numbers, suppose you put that many stones in a bag. I can see the bag, but I do not know how many stones are inside it.

When I say, "Add three," you put three more stones next to the bag. I can now see a bag and three stones. I tell you, "Double." You bring up exactly the same things again. I now see two bags and six stones. "Take away four." You remove four stones. I see two bags and two stones. "Halve." You do so. I now see one bag and one stone. "Take away the number you first thought of." The number you first thought of was the number of stones in the bag. So you remove the bag. That leaves just one stone. The final answer of this trick is always **1** —regardless of the number you may have chosen.

In C1 through C4 below, some are tricks and some are not. In the tricks you can tell a person the answer, because whatever number he thinks of, he will always get the same answer. In the others, this does not happen.

Can you find out, by drawing bags, which are the tricks? It is easy to make mistakes, so test your answers; think of numbers and see whether the same answer always comes.



C1. Think of a number ____________

Add **5** ____________

Multiply by **3** ____________

Subtract **9** ____________

Divide by **3** ____________

Take away the number you first thought of ____________

C2. Think of a number ____________

Add **3** ____________

Double ____________

Subtract **2** ____________

Take away the number you first thought of ____________

C3. Think of a number ....

Add **1** ....

Double ....

Add **10** ....

Divide by **4** ....

Take away the number you first thought of ....

C4. Think of a number ....

Add **3** ....

Double ....

(Continue in next column)

Add the number you first thought of ....

Divide by **3** ....

Take away the number you first thought of ....

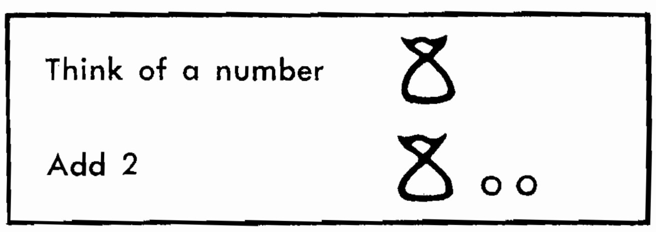
Making Up Your Own Tricks

You will see that there are very many tricks of this kind. How would you make one up for yourself? Before you read further, you may like to see if you can invent your own trick.

What makes a trick of this kind work?

Suppose you say to a friend, "Think of a number. Add **2**." You certainly cannot tell him the answer. If he thought of **5**, his answer would be **7.** If he thought of **10,** his answer would be **12.** If he thought of **1,000,000,** his answer would be **1,000,002.** You have no way of telling what his answer is.

In pictures it would look like this:

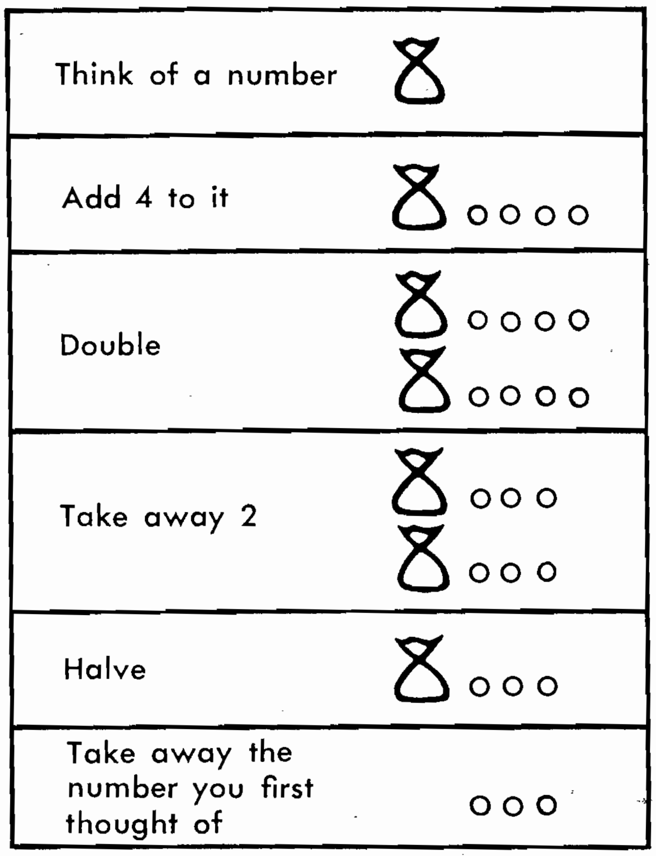


You do not know how many stones are in the bag. That is why you cannot tell the answer. To make a trick, you have to arrange things so that *no bag is left at the end.*

Suppose you say, "Think of a number. Add two. Take away the number you first thought of." The first two steps are shown in the picture above.

In the last step—"Take away the number you first thought of"—you remove the bag. Two stones are left and the answer is always **2.** This is a trick, but a rather feeble one. Many people would see right through it and would not be surprised at all. To surprise your friend, you have to make the trick a bit more complicated to hide what you are really doing.

Here is an example:



The answer is **3**. Here, as you see, no bag remains at the end. The step before the end should leave just one bag. When you say, "Take away the number you first thought of," that gets rid of this bag.

A Plan To Save Labor

You may get tired of drawing bags and stones, particularly if fairly large numbers appear in a trick. It is easy to avoid most of this labor. Instead of drawing two bags and six stones, you can write 2 δ + 6. It would be most wearisome to draw a hundred bags and fifty stones. It is easy to write 100 δ + 50.

(Be careful not to let your bag look like the figure 8.) In this shortened way, write:

Five bags and three stones ....

Ten bags and one stone ....

Three bags and four stones ....

Three bags 3

Four stones 4

Two bags ....

Five stones ....

Two bags and five stones ....

Finding Endings for Tricks

It does not matter how a trick begins, it can always be finished successfully. Here is the beginning of a trick in words and pictures:

Think of a number	δ
Add one	δ + 1
Double	2 δ + 2
Add one	2 δ + 3
Double	4 δ + 6

How are we to finish this trick? We aim to get to a statement that has but one bag, and then to take that bag away. This is how all our earlier tricks ended.

At present we have four bags in the picture. If we divide by 4, that would give one bag. But it is awkward to divide 6 stones by 4. Before we divide, we might add or subtract some number that would make the division by 4 easy to carry out.

There are many ways of finishing this trick. Here is one of them:

We had reached	4 δ + 6
Add **6**	4 δ + 12
Divide by **4**	4 δ + 3
Take away the number you first thought of	3

The answer is **3**

It is wise to make sure that we have not made any slips. Try thinking of different numbers and check to see that the answer 3 does come whatever number you choose.

Try It Yourself

Some beginnings of tricks are given below. Find a good ending for each of them. This booklet cannot give answers because there are many different ways in which these tricks could end. If you get a trick that works, your answer is right.

Endings wanted for these tricks!

D1. Think of a number ....

Add **3** ....

Double ....

Subtract **2** ....

D2. Think of a number ....

Add **5** ....

Double ....

Subtract **6** ....

D3. Think of a number ....

Add **1** ....

Double ....

Add **4** ....

D4. Think of a number ....

Add **1** ....

Double ....

(Continue in next column)

Add **1** ....

Multiply by **3** ....

Add **3** ....

D5. Think of a number ....

Add **1** ....

Double ....

Add **1** ....

Double ....

Subtract **5** ....

Double ....

From Pictures to Shorthand

We have been using the picture of a bag to help us imagine the number somebody thought of. Suppose we erase the top and bottom of the bag. This will change

to  , which looks much like the letter X.

It is much quicker to write X than to draw a bag. Instead of 4  + 6, we now write 4X + 6.

This is a very convenient shorthand. We now have four different ways of showing the same thing:

1. *Words.* Four times the number somebody thought of with six added.

2. *Picture.*  ____________

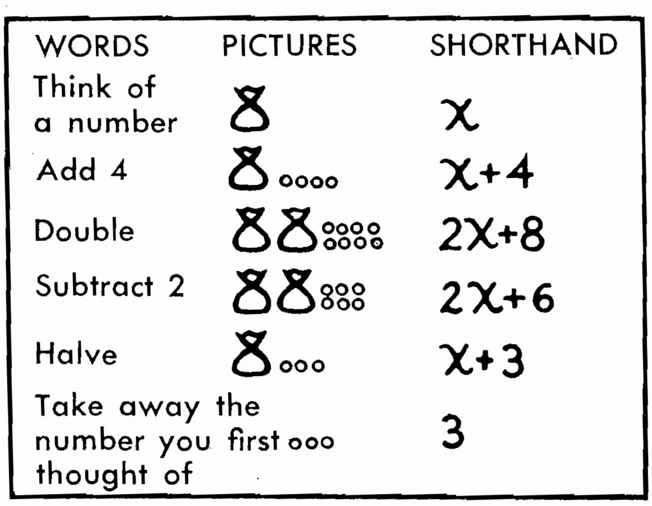
3. *Shortened Picture.* 4  + 6

4. *Shorthand.* 4X + 6

The shorthand form, Number 4 above, is very quick and easy to write. It says just as much as all the words in Number 1. A scientist will often use Number 4, hardly ever Number 1. The pictures are useful to help you imagine what is happening and to see why the trick works.

Some Shorthand Exercises

The illustration below shows the three ways we may tell about the same trick.



Some exercises are given so that you may get used to reading and writing in our shorthand. We take some idea and express it in each of the forms which we shall call #1, #2, #3, and #4. The first exercise has the answers given. You should be able to fill the spaces in the other exercises.

I. #1 Three times the number thought of

#2 ____________ 

#3 ____________ 3 

#4 ____________ 3X

II. #1 Twice the number thought of

#2 ____________

#3 ____________

#4 ____________ 2X

III. #1 Four times the number thought of

#2 ____________

#3 ____________

#4 ____________

IV. #1 Three times the number thought of with two added

#2 ____________

#3 ____________

#4 ____________

V. #1 Twice the number thought of with 3 added

#2 ____________

#3 ____________

#4 ____________

Chapter 4

ALGEBRA, THE LANGUAGE OF SCIENCE

In Chapter 3 we talked about tricks that we can do with numbers and bags of stones. By the end of that chapter, you may have guessed where this was leading. Most people know that X has something to do with algebra. The purpose of Chapter 3 was to introduce you to algebra.

In fact, the whole of Chapter 3 dealt with algebra. If you answered the questions on page 10 by drawing bags and stones, you were doing algebra.

We now want to answer the question, "How does algebra help us to work with and understand science?"

This question will be answered in two parts. First, we shall see how algebra is connected with the guessing game described on pages 6 and 7 of this booklet. Then we will show that science itself is a kind of guessing game.

Do you remember how the guessing game was played by your group? When Henry rose, he had decided that, when someone said a number, he would answer with one more than that number.

The game begins, "Say a number." Our tricks on page 10 began, "Think of a number." Our bag and stone pictures will do just as well for the game as for the x tricks. Instead of putting into the bag the number of stones you have thought of, you put in the number you have said.

If we picture the number you say as "a bag," we must picture Henry's answer as "a bag and a stone."

Now we can translate this into shorthand. If you say X, Henry answers X + 1.

Ann decided to double whatever number was called. If you say a number, she answers twice your number. If we picture your number as "a bag," we must picture her answer as "two bags." In shorthand, if you say X, Ann answers 2X.

Whatever number was said, Joe decided to subtract that number from 1,000.

To the number 5, Joe answered 1,000 − 5
To the number 7, Joe answered 1,000 − 7
To the number 123, Joe answered 1,000 − 123
To the number 911, Joe answered 1,000 − 911
To X, Joe answered 1,000 − X

Jane's rule can be described in two ways. Jane might have thought, "Whatever number is said, I will answer half that number." Equally well, she might have thought, "Whatever number is said, I will divide that number by 2."

If X is short for "the number someone has said," $\frac{1}{2}X$ can be written for "half that number." For "that number divided by 2" we could write $X \div 2$, or $\frac{X}{2}$, or X/2.

In shorthand, then, Jane's rule may be written in any of these ways:

If you say X, Jane answers $\frac{1}{2}X$
If you say X, Jane answers X/2
If you say X, Jane answers $\frac{X}{2}$

On page 7 of this booklet you answered some questions about the guessing game. We have already translated into algebra the rules used by the students in questions 1 through 4.

Translate into algebra your answers to questions 5 through 9. The answers can be found somewhere in the following list:

x	x + 1	x + 2	x + 3	x + 4
x − 1	x − 2	x − 3	x − 4	x − 5
2x	3x	4x	5x	6x
7 − x	8 − x	9 − x	10 − x	11 − x

$\frac{x}{2}$ or $\frac{1}{2}x$	$\frac{x}{3}$ or $\frac{1}{3}x$	$\frac{x}{4}$ or $\frac{1}{4}x$

From this list, choose the correct answers for the spaces below. Refer to questions 5 to 9 on page 7.

Question 5. If you say x, Sue answers ....
Question 6. If you say x, Fred answers ....
Question 7. If you say x, Bill answers ....
Question 8. If you say x, Cathy answers ....
Question 9. If you say x, Jack answers ....

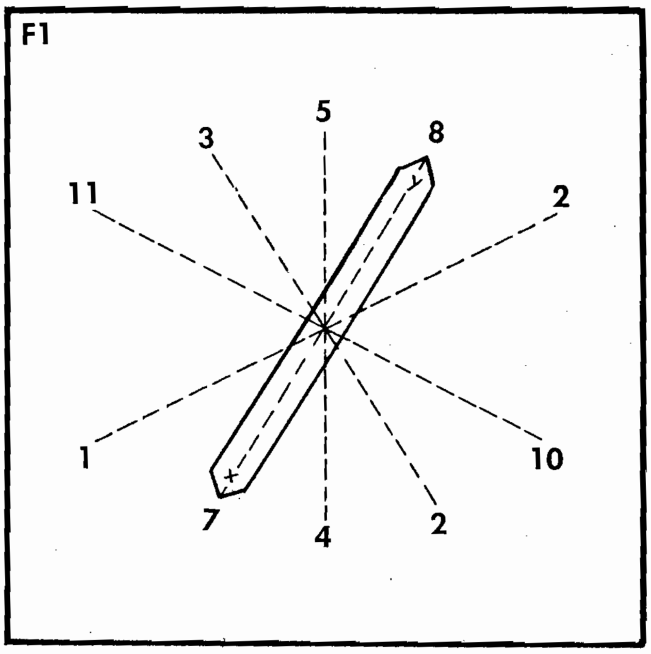
Here is a question that goes the other way round. Mike decides that, if you say **x**, he will say **10x + 3.** If we picture your number as a bag, we must picture Mike's answer as ten bags and three stones.

You say	Mike answers
1	....
2	....
3	....
4	43
5	....
6	....
7	....
8	....

Do you notice anything about the numbers Mike answers? What would be the simplest way for him to find the number he ought to answer?

Guessing Machines

This is another form of the guessing game.



Some numbers are marked on paper or cardboard, as shown in the illustration above. A pointer, with x marked at one end and y at the other, is pinned by a thumbtack at the center so that it can turn freely.

You can think of this as a very simple calculating machine. The question is, "What does it calculate?"

In the machine shown here, when x points at 7, y points at 8. When x points at 10, y points at 11. (The numbers must be carefully arranged in a circle so that the proper numbers are opposite each other.)

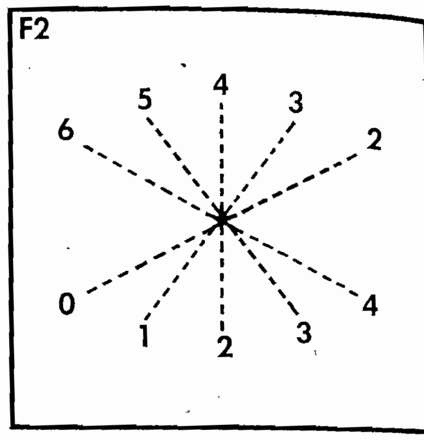
On this machine, whatever number x points at, y points at a number that is one more than that. So this is a machine for adding 1—not a very exciting machine! The law it illustrates is $y = x + 1$, since the number y points at is always found by adding 1 to the number x points at.

Making Other Machines

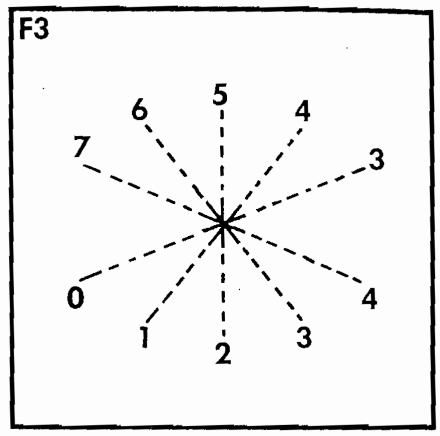
On this and the following pages are 12 different machines. Cut out a pointer just long enough for use on these machines. Mark x on one end of it and y on the other. The same pointer can be used on each of the machines in turn.

Before you begin to study each question, fasten the center of the pointer to the center of the machine by a thumbtack. The end y should always point toward the numbers at the top, the end x to the numbers at the bottom.

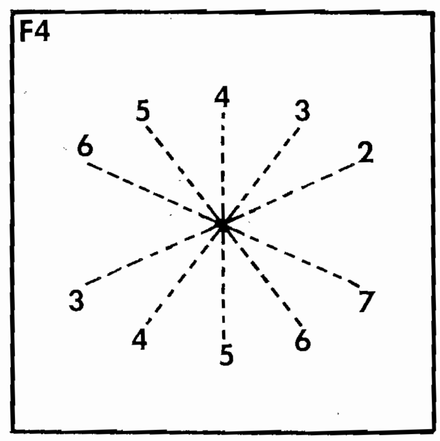
Note the y number opposite each x number. Then, in the space below, write the law belonging to each machine. Use the shorthand form. It is much easier to write $y = 3x$ than to write the whole sentence, "The number y points at is 3 times the number x points at." As soon as you have done each question, check your answer with the answers at the back of the book.



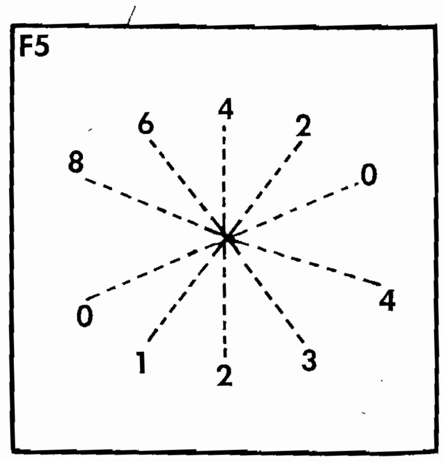
Machine F2
For this machine
$y = x +$ ....



Machine F3
For this machine
$y =$ ....

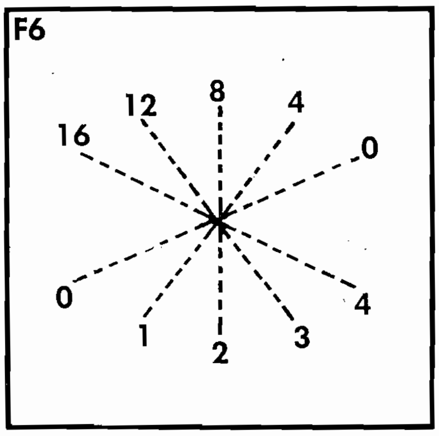


Machine F4
For this machine
$y = x -$ ....

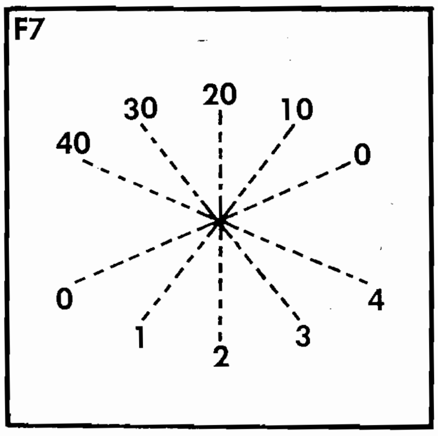


Machine F5
For this machine
$y =$ x

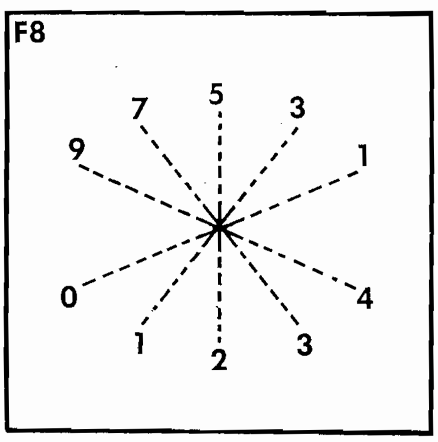
Machine F6
For this machine
y = ... x



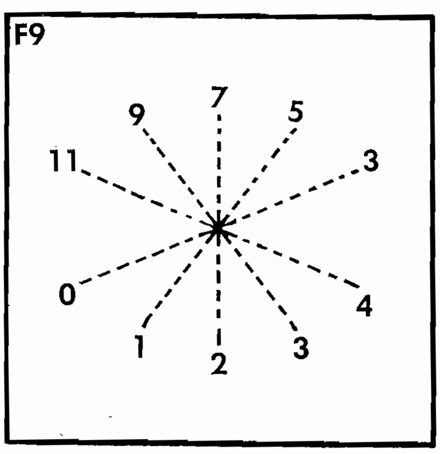
Machine F7
For this machine
y = ... x



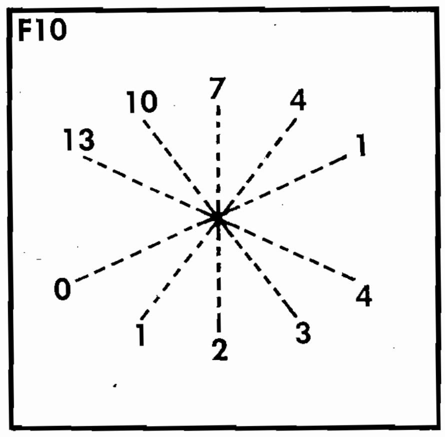
Machine F8
For this machine
y = 2x + ...



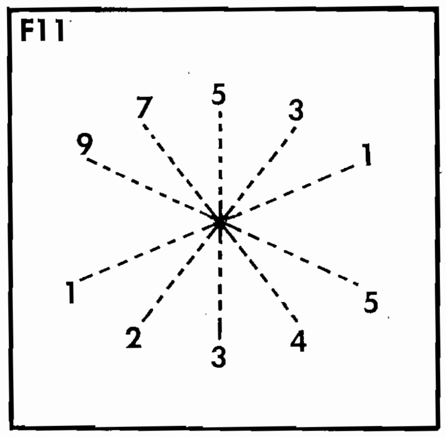
Machine F9
For this machine
y = ...



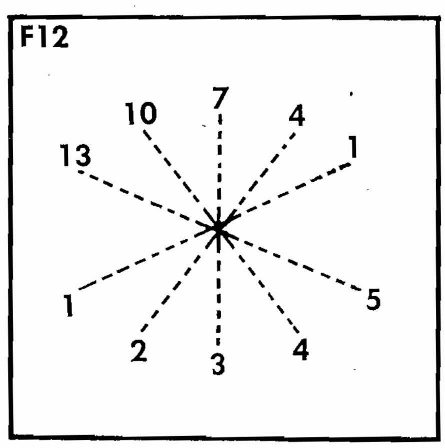
Machine F10
For this machine
y = ...



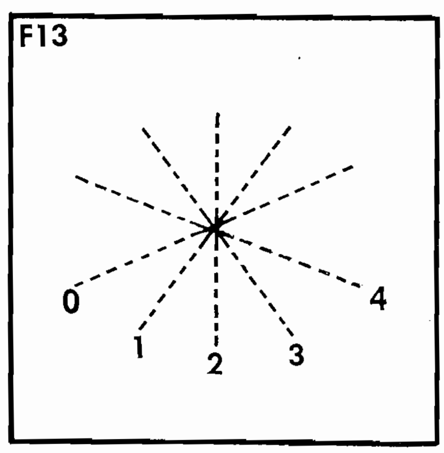
Machine F11
For this machine
y = ...



Machine F12
For this machine
y = ...



Machine F13
For this machine
y = 5x + 4.
Fill in numbers on y side of the machine.



Guessing and Science

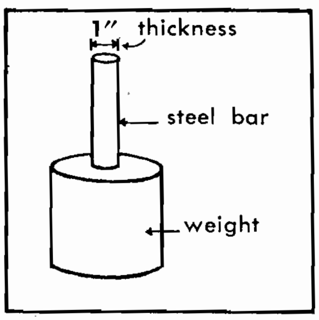
A scientific experiment is much like the guessing games and the guessing machines. In the games, you can call out any number you like and you will receive a definite answer. With the machines, you can choose which number x is to point at; then y will point at a definite number.

It is as if the machine had given you an answer. In an experiment in science, we choose a number, and Nature gives us an answer. Here are some examples:

1. *How long does a stone take to fall to the ground from any height?* We might drop a stone from a height of 50 feet and see how many seconds it took to fall. We might drop other stones from heights of 100 feet, 150 feet, 200 feet, and see how long each took to fall.

Then we would try to find some law for the results. Here we have chosen the heights for our experiment, but we have to get the answer in time from Nature.

2. *What weight will a round steel bar support?* We might take a steel bar 0.1 inch thick and see what weight hung on the end would break it. Then we might try steel bars 0.2 inch, 0.3 inch, and 0.4 inch thick, and see what weight was required to break each.



We would hope to find some simple law behind these results. Here we choose the thickness of the bar, and the experiment replies with the weight that will break it.

We shall think more about this question when we read page 19.

3. *How far does a car go after the brakes are applied before it stops?* We might drive a car at 10 miles per hour, put the brakes on when it crossed a certain line, and see how far it went before coming to rest. We might repeat the experiment with the car going at 20 miles per hour, then at 30 miles per hour, then at 40 miles per hour. We would look for a pattern or law in the results.

The results as given in the laws of one state are found on the next page.

Here we choose the speed of the car, and that determines the braking distance. (The condition of the road surface must be the same for each experiment, of course.)

4. *By what law does an animal grow?* We might take a baby mouse and weigh it when it was one day old, when it was 2 days old, when it was 3 days old, and so on.

Living things are more complicated than machines, and we would not expect to find a very simple law. Still, we might make some discovery.

Here we choose the ages 1 day, 2 days, 3 days, etc. The experiment of weighing the mouse gives an answer to go with each of these.

A Shorthand for Science

Earlier we used **x** as short for "the number someone thought of." A student might say, "I do not like **x** because it does not help me to remember. I would rather use **n**, because that is the first letter of *number*, and it helps me to remember that **n** stands for the *number thought of*.

This student has every right to choose **n** rather than **x**. It does not matter what letter you use, so long as you know what it means.

In the guessing game, a student might use **c** as short for "the number someone calls out," and **a** as short for "the number given as an *answer*." Henry's rule could be written $a = c + 1$, because Henry answers one more than the number called out.

In scientific work, this arrangement is very convenient. In our experiment about the falling stones, we could use **x** as short for the number of feet through which the stone falls and **y** as short for the number of seconds it takes to fall that distance.

There would be nothing wrong with doing this if you wanted to. But most people seem to prefer the following plan.

The stone is dropped from a certain *height*, so we use **h** to stand for the number of feet in this height. It takes a certain *time* to fall, so we use **t** for the number of seconds it takes to fall.

We want to find a rule connecting **h** and **t**. This rule will be discussed a little later.

Letters Stand for Numbers

Notice, by the way, that **h** and **t** in the falling stone experiment are both *numbers*. Do not think of **h** as "the height" and **t** as "the time." We shall use **h** to mean "the number of feet in the height" and **t** to mean the "number of seconds in the fall." For example, a stone falls from a height of 64 feet in a time of 2 seconds. We write this $h = 64$, $t = 2$.

Engineers sometimes use a different kind of algebra in which they write "$h = 64$ feet" and "$t = 2$ seconds," but it is not wise for one who is just beginning algebra to use such a system. Both in higher mathematics and in engineering there are places where letters stand for things that are not numbers—but you can learn about that when you reach that stage.

In this book, letters such as **x, y, h, t,** *will always stand for numbers.*

In our second experiment above, we know how thick a steel bar is, and we discover by trial what weight will break it. Suppose we find that a weight of 1,000 pounds will break a steel bar 0.1 inch thick. We might write $w = 1{,}000$ because **w** reminds us of *weight*, or $p = 1{,}000$ because **p** is the number of pounds, or $b = 1{,}000$ because it is something to do with breaking. It does not matter which of these you use, *provided you say what you are doing.*

Always begin your work by explaining the code you are using. Anyone else who reads your work will then be able to understand what you are doing. You yourself may want to refer to it weeks or months later. You will have forgotten what code you used, and will find yourself asking, "What does **w** stand for? Did I use it because it reminded me of weight, or because it had something to do with the width of the bar?"

In the same way, the numbers that measure the *thickness* of the bar might be indicated by **t**, or by **b** for *breadth*, or by **d** for *distance* or *diameter*,* or by **i** because it is measured in *inches*. You might prefer some other letter. Whatever letter you choose, show clearly what it means.

The records of this experiment might appear like this:

w = the number of pounds weight needed to break a round steel bar.

d = the number of inches in the diameter of the bar.

d	w
0.1	1,000
0.2	4,000
0.3	9,000
0.4	16,000
0.5	25,000
0.6	----------
0.7	----------

We have not yet learned how to write the rule connecting **d** and **w** in shorthand. This we shall study later. But can you guess what numbers should go in the spaces above?

Note: Experiments never work out as neatly as this in practice! Errors of measurement always creep in.

The numbers in the above table were not in fact taken from an actual experiment. It is very unlikely that the weight needed to break the first bar would turn out to be a nice convenient number like 1,000. In engineering, the most awkward numbers happen all the time. The questions you find in books are always specially simple ones. Otherwise you would get so tied up in the details of arithmetic that you would not notice the important things, the simple laws behind the complicated numbers.

The numbers in the table above have been simplified, but the law is real engineering. The strength of a steel bar really does grow in the way the numbers suggest.

In the table you will see that doubling the thickness of the bar does *not* simply double the strength; it multiplies it by four. Look, for example, at the top two rows. Changing **d** from 0.1 to 0.2 doubles the thickness but multiplies the strength by 4; **w** changes from 1,000 to 4,000.

Again, if you change the diameter of the bar from 0.2 to 0.4, you have doubled the diameter, and the table shows that the weight the bar can support is multiplied by 4.

This effect is actually found in practice. This principle would be used by an engineer in designing a structure. For example, it would be very important when the engineer was deciding how big to make the supports of a great bridge.

The whole of Chapter 6 in this book is devoted to examples of this law. By reading these examples you may be able to pick up a hint as to why the strength of a steel bar is related to its thickness in this particular way.

If you were making ropes by putting many pieces of string together in a bundle, you would find that it would take four times as many strings to make a rope 2 inches thick as it would to make a rope one inch thick. The same law is at work here.

*Diameter is the scientific word for the distance across a circle.

Car Speed and Brakes

In the state of Connecticut, the following law applies to cars with 2-wheel brakes. When traveling at 20 miles per hour, the car must be able to stop within 40 feet. If going at 30 miles per hour, it must stop in 90 feet. If going at 40 miles per hour, it must stop in 160 feet. The law says that if the brakes on a car will not stop the car in those distances the brakes are defective and must be repaired.

We could write these numbers in a table like this:

Car's Speed in m.p.h. v	Braking Distance in feet s
20	40
30	90
40	160
50	------
60	------

Can you fill in the spaces?

The law connecting **v** and **s** is one that we will learn to write later on.

You may wonder why we have used **v** for speed and **s** for distance, because **v** and **s** do not seem to fit these words at all. The reason is that these are the letters you will meet in scientific books. They have been used for about 300 years, ever since the laws of mechanics were first discovered.

At that time, scientific books were written in Latin, and the letters were suggested by Latin words. You can understand the use of **v** when you know that the Latin gives us our word "velocity." The **s** was suggested by the Latin word for "space."

You will be interested to compare the numbers in this experiment with those in the first column. There is a pattern here which we shall see more clearly when we read Chapter 6.

SOME SIMPLE LAWS

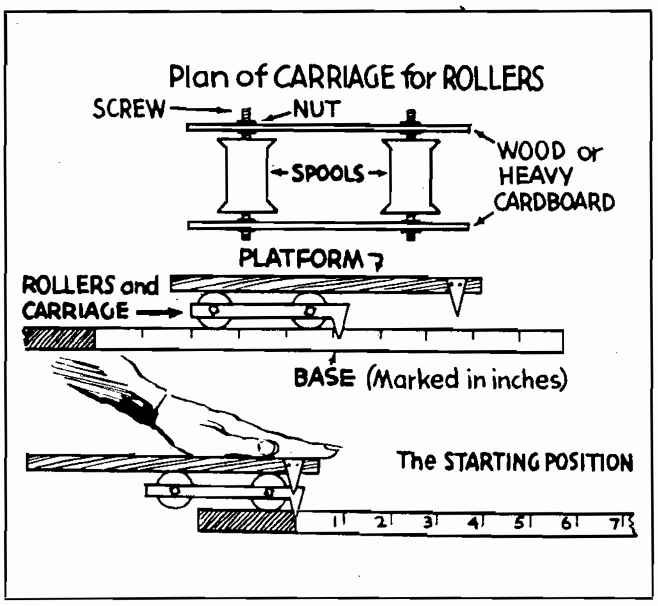
Chapter 5

In the previous chapter we saw several simple examples of the way algebra helps us to understand science. In experiments we arrived at pairs of numbers which were in a pattern. The task of the scientist is to find the pattern in the numbers.

In the present chapter we shall suggest several experiments from which you may be able to find laws of science.

The Platform on Rollers

This experiment requires that you make a piece of apparatus. However, only junk is needed to make the apparatus. A study of the pictures below will show how the junk is put together.



For the rollers you can use cotton spools or dowelling. The platform and base are simply flat pieces of wood.

Put the rollers and the platform in the starting position as shown in the picture. Both pointers are opposite zero.

With your hand flat on the platform, push forward until the pointer fastened to the rollers is opposite 1. Where is the other pointer? The other pointer shows how far the platform has gone.

Push the rollers another inch forward, and see where the platform is now. Continue in this way for several more inches and enter your results in the table at the top of the next column.

As you push the platform forward, you find that it moves faster than the rollers.

In this experiment, we shall use **r** to stand for the number of inches the rollers have gone. We shall use **p** to stand for the number of inches the platform has gone.

Number of inches rollers have gone	Number of inches platform has gone
r	p
0	0
1	....
2	....
3	....

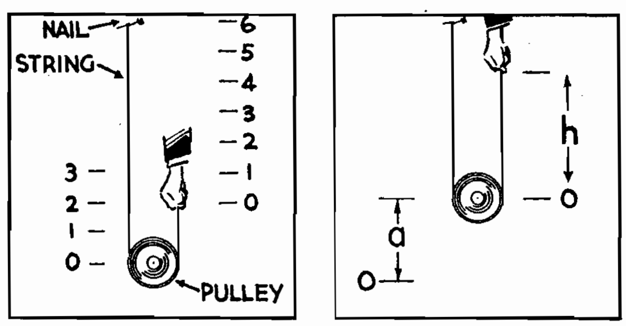
What law or rule can you see in the numbers above? Fill in the space in the sentence below. Just a number is needed.

The number of inches the platform goes is .... times the number of inches the rollers go.

If you write the same number in the space below, you have this law in the shorthand of algebra.

p = ... **r**

The Pulley



This is another very simple experiment. Mark a chalkboard with numbers as shown. The divisions could be, say, 4 inches each.

The actual length does not matter so long as the divisions are all the same.

If you do not have a pulley, you can use your thumb for the string to go around. Equally well you could use a pencil or a bottle—any small object that the string can slip around. You will have to use your left hand to keep the object you use for a pulley from falling.

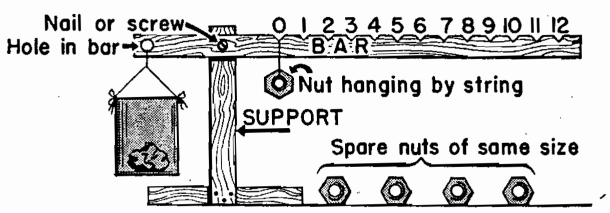
As the hand pulls up on the string, both the hand and the pulley rise. But they do not rise through equal distances. Try it and see. Checking by the numbers on the chalkboard, fill in the spaces in the table that follows at the top of the next page.

Number of divisions the pulley has risen **a** (for short)	Number of divisions the hand has risen **h** (for short)
0	0
1	....
2	....
3	....

The rule is **h** = ...**a**

Note. You may wonder why **a** was used to show how far the pulley went. The reason is this. In the question before, **p** was used for the distance the platform went. As "pulley" also begins with "P," we might have used **p** again in this question for the number of divisions the pulley has risen. Some readers might find it confusing to use the same letter with two different meanings so close together. We use **a** because the pulley can rise any number you like to mention.

Home-Made Weighing Machine



The support can be made in any way you like. Wood or a metal bracket can be used. The bar must turn freely about the rail.

If you use a long bar, you can have more notches than the picture shows. A convenient size is to have the notches one inch apart and the hole in the bar four inches from the rail.*

It is important that the can should hang from the bar by a single string as shown. There is a package of old nails, screws, bolts, etc., in the can. You must put just enough in this package to make the bar balance with the nut in notch **0**. Then put one nut in the can. Find how many notches the hanging nut must slide along the bar to restore the balance. Next, put another nut in the can, and so on. Of course all the nuts must be identical with the nut that hangs by a string. Fill in the table below.

Number of nuts in can w for short	Number of notch giving balance n for short
0	0
1	....
2	....
3	....
4	....

What is the rule? **n** = ... **w**

*The answer at the back of the book is for a machine of this size. With different distances, you will get a different law.

The Bouncing Ball

I stood near a brick wall and dropped a new tennis ball. I used the thickness of the bricks as a measure. I noticed the height (**h** bricks) from which the ball was dropped, and the height (**b** bricks) to which it bounced. I did not expect to get very exact results. Probably I did not catch the ball when it was exactly at the top of its bounce.

Anyhow, here are the results I got.

h	b
0	0
4	$2\frac{1}{4}$
8	$4\frac{1}{2}$
9	5
12	$6\frac{1}{2}$
20	$10\frac{1}{2}$
28	14
30	15

The results for heights 0, 28, and 30 suggest that the bounce is just half the original height: **b** = $\frac{1}{2}$**h.** Look at the other numbers and see if you think that this rule is fairly near to the truth.

Nearly always, when we make measurements, errors creep in. Do not expect to find a simple law that fits all observations exactly.

See if you get results like these with a bouncing ball. You may get different laws with a new ball and an old one. The hardness of the ground will also make a difference.

Two Notes on Algebra

1. In arithmetic, we use the sign × for multiplication. In algebra, this sign is not convenient because it can easily be mistaken for x.

If you will look back through our work in algebra, you will find that *we have never used any multiplication sign at all.*

On page 13, for example, we used 3x for "three times the number we thought of." We did not put any sign at all for *times.*

This was natural. We used "a bag" to picture "the number of things thought of." For three times this number, we drew a picture of three bags. We say "three bags"; we do not say "three times a bag." In our shorthand we do the same thing. We write 3x. We do not write 3 × x, as a rule.

In algebra, when we write **5a** or **2h** or **7x,** with no sign at all between the number and the letter, remember that each of these represents multiplication. If **a** stands for any number, **5a** stands for 5 times that number.

We have used this way of writing in our experiments above. The platform goes 2 *times* as far as the rollers; **p** = **2r.** If the pulley rises any number of divisions, the hand rises twice as many; **h** = **2a.** The number of the notch is 4 *times* the number of nuts being weighed; **n** = **4w.**

Sometimes a dot is used to show multiplication. **y** = **10 • x,** then, means y is 10 times x.

2. *The simplest law of all*—it is a strange thing that some questions are hard to answer because they are too simple. In the guessing game, you may be able to write the laws **y = 2x** and **y = x + 3**, and yet find difficulty with the following:

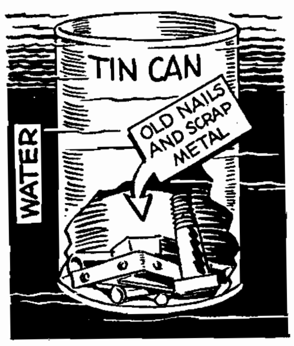
Number Called	Number Answered
x	y
5	5
2	2
7	7
3	3
11	....
4	....

You will be able to fill in the spaces above. The number answered is the *same as* the number called. But how shall we write this law in our shorthand?

We write **y = x**

The Law of Floating

This law is used in the design and making of ships, rafts, pontoon bridges, and balloons.



Get some tin cans of different sizes. Fill each can with water, weigh it, and write down the weight.

Empty the cans and let them dry. Put old nails or scrap metal into one of the cans, and float the can in a bowl of water. Keep adding metal (without tipping the can) until the can is on the point of sinking, as shown in the picture.

Remove the can, dry it, and weigh it with the metal still in it.

Do the same for another can. Try the same experiment with several cans.

What do you learn about the two weights—the weight of the can full of water, and the weight of the can with enough metal to make it almost sink?

....

....

....

Code:

W = the number of ounces the can full of water weighs

M = the number of ounces the can with its load of metal weighs.

Which of the following laws agrees best with the results of your experiments? (Of course, you expect some errors of measurement.) None of these laws will fit your results exactly.

(1) W = M + 5   **(2) W = 2M**

(3) M = 2W   **(4) M = W**

(5) W = 2M + 3

The law seems to be ....

....

The law you have just discovered here is known as the Principle of Archimedes. Archimedes (är′kĭ-mē′dēz) was a famous Greek mathematician who lived more than 2,000 years ago.

Power for Airplane Engines

The following information about airplane engines was taken from the *Encyclopaedia Britannica.* It tells us the weight (in thousands of pounds) that various airplanes can carry, and it also tells us the power of the engine in each plane (in hundreds of horsepower). We should use **P** for the power and **W** for the weight.

Airplane	Power P in hundreds of horsepower	Total Weight W in thousands of pounds
DC-3	24	24
DC-6	84	95
Convair 240	42	40
Grumman G73	12	12½
Beech Model k8S	9	9
Lockheed Constellation	88	94
Boeing Stratocruiser	140	142½

There is no simple law that connects these numbers *exactly,* but they do come quite close to a very simple law. This law tells you about how much power to provide for an airplane of known weight.

Which of the laws below do you think is most suitable for this purpose?

(1) P = 2W   **(2) W = 2P**

(3) P = W − 10   **(4) P = W + 10**

(5) P = W   **(6) P = 2W + 5**

The most suitable law is ....

If you were designing an airplane similar to one of the types above, and this plane was to weigh 70,000 pounds fully loaded, about how much power would you expect it to require? ....

Chapter 6

AN IMPORTANT LAW

You may sometimes think, "Why learn math if I am interested in science?" There are many different kinds of science—astronomy, biology, chemistry, electricity, gravitation, heat, light, sound, magnetism, mechanics—to mention but a few. If you are interested in electricity, why not learn just the calculations needed in electricity? Why bother about math?

This may sound reasonable, but in fact it would be extremely wasteful. Look at the following list of questions. Do not be disturbed if you cannot answer them. The point is to see if you can discover any connection between them, even though they seem very different.

1. How high must a cliff be for a person on it to see 12 miles out to sea?

2. What shape is the jet of water in a drinking fountain?

3. How much floor covering do you need for a square room 10 feet by 10 feet?

4. In what path does a comet move around the sun?

5. If you double the length of a pendulum, do you double the time it takes for its swing?

6. What is the best shape for the mirror in a reflecting telescope?

7. How does the strength of a steel bar depend on its thickness?

8. How does the heat produced by an electric heater depend on the current passing through it?

9. At what rate does the illumination fall off, as you move away from a lamp?

10. How does the pull of the earth decrease as you move away into space?

11. How does the pressure of the wind on a house depend on the speed of the wind?

12. If you double the width of a can, what effect does that have on how much water it will hold?

13. How does the braking distance of a car depend on its speed?

14. By what law does a stone fall?

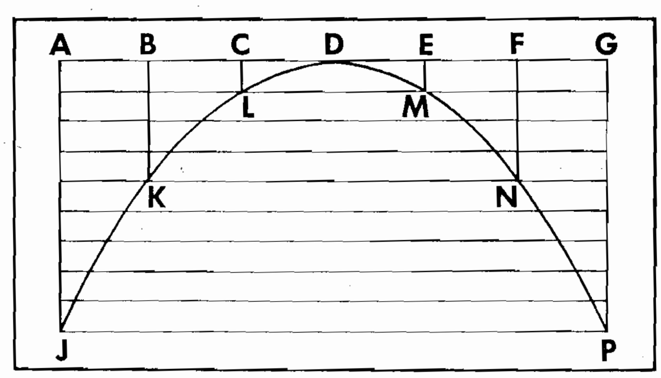
15. If you could enlarge a flea to the size of an elephant, would it still jump as well?

These questions deal with many different things, yet all of them depend on one simple mathematical pattern—the pattern of the numbers 0, 1, 4, 9, 16, 25, ________.

Look through this booklet again from the beginning. How many places can you find where these numbers, or something like them, came in? Write in the spaces below the places where these numbers appeared, and what subject they had to do with.

Page	Subject
________	________________________________
________	________________________________
________	________________________________
________	________________________________
________	________________________________

A Jet of Water



You can draw the shape of a jet of water in this way.

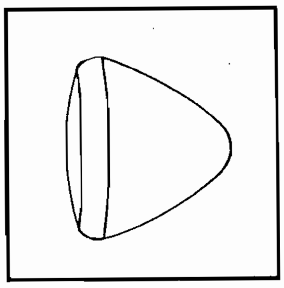
Take some graph paper, and mark the points A, B, C, D, E, F, G. These are evenly spaced. You can choose the distance between them for yourself.

From these points we draw lines straight down. EM and CL are 1 inch long. FN and BK are 4 inches long. AJ and GP are 9 inches long. Connect the points J, K, L, M, N, P with a curve as shown.

If you fasten a hose to a water faucet, you should be able to make a jet of water run in the curve you have just drawn. You will need to adjust the faucet until the water comes out at the right speed. If you have chosen a large distance between the points A and B, the water will have to come out faster for the jet to fit your curve. If you have A close to B, your curve will fit a jet squirted at a low speed. You can test your curve by holding it close to the jet of water.

We have now answered Question 2 in our list.

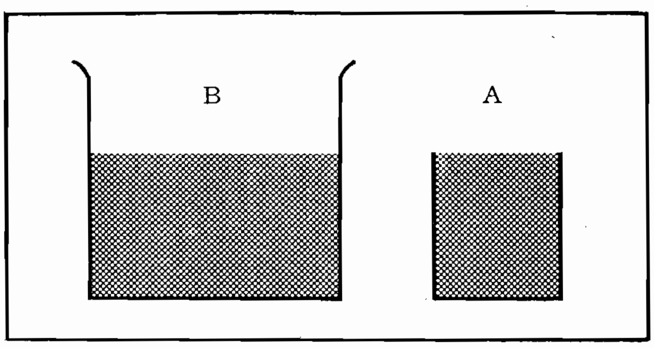
The curve we have just drawn is called a parabola (pă-răb'ô-lȧ). A comet passing the sun often moves in



The curve of the reflector of an automobile headlight is the same as the curve made by the jet of water.

a curve of this kind (Question 4). This curve is also used for the mirror of a reflecting telescope (Question 6). You may have noticed that the reflectors of automobile headlights use this shape.

Cans of Water



Now for an experiment using cans of water in a different way. Find two cans, one of which has exactly twice the width of the other.

In the picture, can B is twice as wide as can A. (For can B, I used a large can that had contained pineapple juice. It was about 7 inches high and 4 inches wide. Can A had contained frozen concentrated orange juice. It was about 4 inches high and 2 inches wide.)

The question is, "How many times must you fill can A with water and empty it into can B, to make the water in B stand as high as the top of can A?"

Answer: ....

It is difficult to find cans just the right sizes for this experiment. If you like, you can use sand instead of water, and make containers out of cardboard.

If you are willing to take some trouble, you can make a whole series of containers. They should all be the same height. The first, A, can be any width you find convenient. The second, B, must be twice as wide as A. The third, C, must be 3 times as wide as A. If you make a fourth, D, it should be 4 times as wide as A.

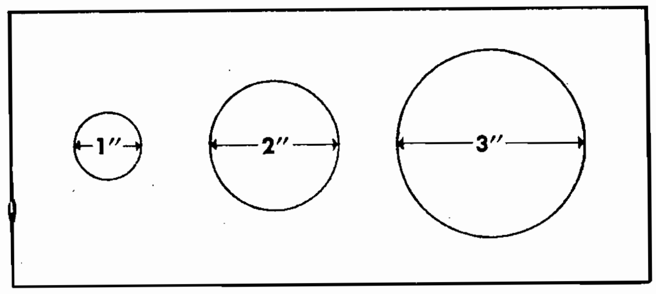
How many times can you empty A into each of the other containers? The answers are all whole numbers.

B is twice as wide as A. It holds .... times as much as A.

C is 3 times as wide as A. It holds .... times as much as A.

D is 4 times as wide as A. It holds .... times as much as A.

Pennies in Circles



On a piece of paper draw a circle of a radius of 1 inch, another of radius 2 inches, and another of radius 3 inches.

How many pennies can you put inside each circle? Each penny must lie flat on the paper, and no penny must stick out over the circle it is in.

You can put .... pennies inside 1 inch circle.

You can put .... pennies inside 2 inch circle.

You can put .... pennies inside 3 inch circle.

Is it true that, if you double the width of a circle, you double the number of pennies you can put in it? ....
If not, is the answer more than double or less than double? ....

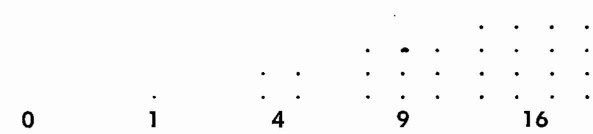
If you multiply the width of the circle by 3, do you multiply the number of pennies it can hold by 3? ....

If not, will it hold more or less than this? ....

Putting pennies into a circle, a lot of space is left between the pennies, and different students may pack the pennies in different ways. The answers to these experiments may not be the same each time the experiment is done. But these experiments show roughly how much more room there is in a circle when its width is multiplied by 2 or by 3.

The Square Law

The numbers 0, 1, 4, 9, 16, ... are called the *square numbers*. You can see the reason for this name:

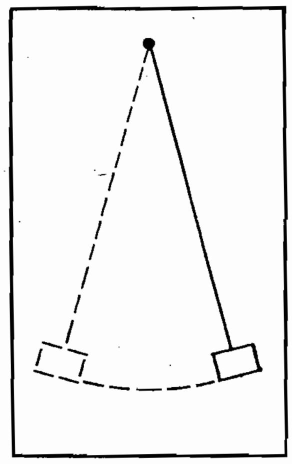


We met these numbers at the beginning of Chapter 2 when we worked out 0×0, 1×1, 2×2, 3×3, 4×4. These numbers have something to do with all the questions given in the long list early in this chapter.

If you look back to page 19, you may see that they have something to do with the strength of a steel bar (Question 7). If you double the width of the bar, you multiply its strength by 4. If you treble the width of the bar, you multiply its strength by 9.

On page 19, you may see that the square numbers also come into the law for the braking distance of a car (Question 13).

The Pendulum



The fact that a swinging weight will go back and forth according to a definite law is used in making many clocks.

It is very easy to make a pendulum. You need only tie a **small** object, such as a metal washer or nut, to the end of a thread, and hang this thread from a nail. Then set it to swinging.

It is hard to time a single swing of a pendulum. If you allow it to make 20 to 30 swings, and see how many seconds this takes, you can work out the time of a single swing quite accurately.

Does it make any difference if a heavy nut or a light nut is tied to the end of the thread? Make several pendulums, using the same length of thread, but with different weights on the end. Time 30 swings of each of these. Record what you observe.

Pendulum	Weight of Nut	Thread Length	Time for Swings
No. 1	....	....	....
No. 2	....	....	....
No. 3	....	....	....

What do you observe? ....

....

....

Conclusion: If the length of the pendulum is kept fixed, increasing the weight of the nut makes the time of swing

longer? ....

shorter? ....

leaves it about the same? ....

Now we study the effect of changing the length of the thread. Use the same nut, but change the length of the thread by which it hangs. See how long it takes to make 30 swings and note the results.

Length of Thread	Time for 30 Swings
....	....
....	....
....	....
....	....

Conclusion: If you make the thread longer, the time the pendulum takes to swing

gets longer? ....

gets shorter? ....

stays the same? ....

Now we are ready to look for the actual law. Take any of your pendulums and see how long it takes to swing. Now try to make one that requires **exactly twice that time.**

The answer is simple. No fractions come into it—only a whole number.

Do you have to multiply the length of the thread by 2? by 3? by 4? by 5? by what? Or perhaps you must divide the length by 2, or 3, or 4?

What would you do to make a pendulum that needs 3 times as long to swing? Experiment with this question. The answer uses only whole numbers.

Conclusions:

To make the swing of a pendulum take twice as much time you have to .... the length of the thread by ....

To make the swing of a pendulum take three times as long, you have to .... the length by ....

If your school is a high building, you may find it possible to hang a very long pendulum to some place. Before you fix it up, try to work out how long it will take to swing. Then, test your guess by seeing what it actually does.

Writing Squares and Other Numbers

As you have seen, the square numbers arise in all kinds of scientific applications. We should like to have a way of writing the square law in the shorthand of algebra. And there is a way of writing it. To explain it, we go back to a question in arithmetic.

Suppose we choose any number, say 2. We can multiply 2 by itself. $\mathbf{2 \times 2 = 4}$. We can multiply the result by 2 again. $\mathbf{2 \times 4 = 8}$. If we like, we can multiply by 2 again and find $\mathbf{2 \times 8 = 16}$.

If we want to show that these numbers came through multiplying by 2 again and again, we can write:

$$4 = 2 \times 2$$

$$8 = 2 \times 2 \times 2$$

$$16 = 2 \times 2 \times 2 \times 2$$

Fill the spaces below so as to continue this pattern:

$$\_\_ = 2 \times 2 \times 2 \times 2 \times 2$$

$$64 = 2 \times 2 \times \_\_\_\_\_\_$$

$$\_\_ = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

If you go on long enough like this, you will come to results like:

$$512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

In the same way, you could start with 3 and get:

$$9 = 3 \times 3$$

$$27 = 3 \times 3 \times 3$$

$$\_\_\_\_ = 3 \times 3 \times 3 \times 3$$

$$243 = 3 \times 3 \times 3 \times 3 \times 3$$

$$729 = 3 \times 3 \times \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

People get tired of writing these long strings of numbers, so naturally mathematicians have tried to find some shorter way of writing these things.

Suppose we try to invent a way of doing this.

You want to tell a friend that you are thinking of $2 \times 2 \times 2 \times 2 \times 2$. Really there are three things that you want to say:

1. You want to tell him the kind of thing you are doing. You are choosing any number and you are multiplying by the same number.

This tells him something, but not enough. You may be thinking of $2 \times 2 \times 2 \times 2 \times 2$ or $3 \times 3 \times 3$ or $7 \times 7 \times 7 \times 7$. He does not know which.

2. You must tell him what number you have chosen, 2. Now he is much nearer. He knows that you are thinking of 2×2 or $2 \times 2 \times 2$ or $2 \times 2 \times 2 \times 2$ or something like these. You still have to tell him when he is to stop multiplying by 2.

3. The third piece of information tells him to write 2 down 5 *times:* 2 2 2 2 2, and put multiplication signs in between. Now he has your number exactly: $2 \times 2 \times 2 \times 2 \times 2$.

So your message has to say (1) keep multiplying, (2) use the number 2, (3) write it 5 times.

A mathematician can give this message in a very short form. He just writes 2^5.

How does this give the message?

The 2 shows that we have to keep writing the number 2. The 5 "up in the air" shows that 2 has to be written 5 times, like this 2 2 2 2 2.

How do we know that multiplication signs must be written between these numbers? That is simply an agreement. Whenever you see a sign like 2^5 or 3^4 or 10^6, it is understood that we are *multiplying* again and again. This type of sign is used *only for repeated multiplication.*

So, 2^5 means $2 \times 2 \times 2 \times 2 \times 2$.

The same code can be used with other numbers.

Example 1. What is 3^4? We have to keep writing 3. How many times do we write it? The number "in the air" says, "Write it 4 times." Now we have 3 3 3 3. To complete it, we write multiplication signs between. Why? Because we always put multiplication signs when we are dealing with something like 3^4.

So we get $3 \times 3 \times 3 \times 3$.

Example 2. What is 10^6? We have to write 10 again and again. How often? 6 times: 10 10 10 10 10 10. Put multiplication signs between: $10 \times 10 \times 10 \times 10 \times 10 \times 10$.

Here are some "code names" for numbers:

2^7; 3^6; 10^3; 5^2; 7^3; 10^2.

Can you fit each of these to one of the numbers below?

10×10	$7 \times 7 \times 7$
$3 \times 3 \times 3 \times 3 \times 3 \times 3$	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
5×5	$10 \times 10 \times 10$

Code names like 2^5 and 3^4 are important in science. They have many applications besides those discussed in this book.

Few students learn this code all at once. Many students fall into the following error. They see 3^4. They think, "This has something to do with four threes. So it must mean 12!"

It does *not* mean 12. For sure, 12 has something to do with four threes. Twelve comes when you *add* four threes. $12 = 3 + 3 + 3 + 3$. But 3^4 comes when you multiply three by itself four times. 3^4 means 3 times 3 times 3 times 3, which is 81.

$$3^4 = 3 \times 3 \times 3 \times 3 = 81.$$

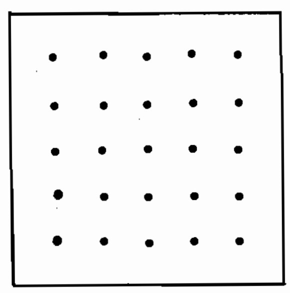
You should practice with this code until you know it thoroughly.

The answers to the questions below are all in the following list: **8, 9, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 243, 1000.**

Can you find which number is the answer to each of the following questions?

$2^3 =$ ____	$5^2 =$ ____	$7^2 =$ ____
$2^5 =$ ____	$10^3 =$ ____	
$3^4 =$ ____	$3^3 =$ ____	
$3^2 =$ ____	$6^2 =$ ____	
$11^2 =$ ____	$8^2 =$ ____	$10^2 =$ ____
$5^3 =$ ____	$3^5 =$ ____	

The Squares



This picture shows the square number 25. We have 5 rows with 5 dots in each. So the number of dots is 5×5.

This number we can write in our code. In 5×5, the number chosen is 5, and it is written *twice.* So 5^2 is the code name for 5×5.

$$5^2 = 5 \times 5 = 25$$

5^2 is usually read "5 squared."

We can write all the square numbers in this way.

$$0^2 = 0 \times 0 = 0$$

$$1^2 = 1 \times 1 = 1$$

$$2^2 = 2 \times 2 = 4$$

$$3^2 = 3 \times 3 = 9$$

$$4^2 = 4 \times 4 = 16$$

$5^2 = 5 \times 5 = 25$, **and so on.**

Fill the spaces below:

.... $= 6 \times 6 = 36$

.... $= 7 \times 7 = 49$

$8^2 =$ .... $= 64$

.... $= 9 \times 9 =$ ....

.... $=$ .... $= 100$

We can write 3^2 and 4^2 and 5^2. Is there anything special about 3, 4, and 5? No. You can choose any number you like. If you say 17, we can write 17^2, which means 17×17. If you say 23, we can write 23^2, which means 23×23.

We could play this as a guessing game. Whatever number you call out, I multiply that number by itself. We could write it like this:

You call out	I answer
3	3^2
4	4^2
5	5^2
17	17^2
23	23^2

We can use x for "the number you call out" and write the rule like this:

You call out	I answer
x	x^2

This means: you call out any number you like; I answer that number multiplied by itself. This is what Nancy was doing in the game on page 7.

We might put it this way. "You call out x; I answer x times x." In this sentence x stands for any number you choose. If you choose 3, we must put 3 wherever x comes, like this:

You call out x; I answer **x times x.**

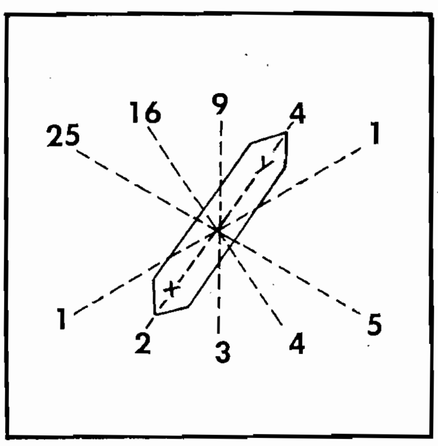
You call out 3; I answer **3 times 3.**

Of course, you do not have to choose 3. You may choose any number you like. Wherever x comes, you erase x and write the number you have chosen. If you choose 5, for example, it will be like this:

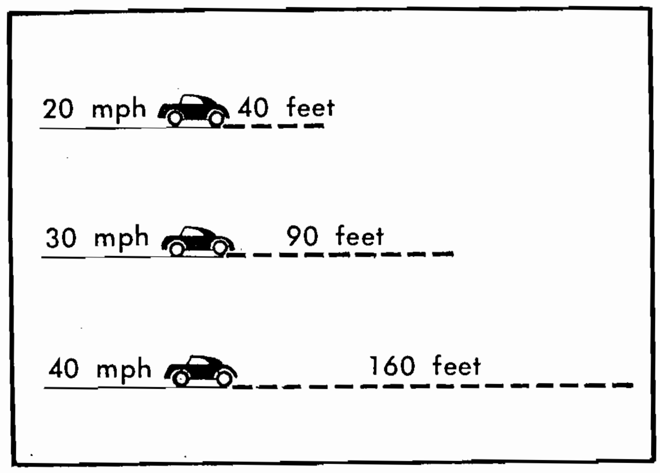
You call out x; I answer **x times x.**

You call out 5; I answer **5 times 5.**

We could make a machine to show this.



The Braking Car



On page 19 we gave some Connecticut regulations for brakes on cars. Here they are:

V	S
20	40
30	90
40	160

The numbers 40, 90, 160 remind us of 4, 9, 16, which are square numbers. So perhaps squares have something to do with this table. You get a square number when you multiply any number by itself. Let us take the numbers under V and multiply them by themselves.

V	V × V	S
20	$20 \times 20 = 400$	40
30	$30 \times 30 = 900$	90
40	$40 \times 40 = 1600$	160

The numbers in the middle are not quite the same as those under *S*, but you will see a pattern. If we divide the numbers in the middle by 10, we get the numbers for *S*.

The Connecticut traffic officers seem to use the following rule:

See how many miles per hour the car is doing.

Multiply this number by itself.

Divide by 10.

That gives you the number of feet the car should need to stop.

We have shorthand for the number of miles per hour the car is doing. We call it **V.** So the rule is, find **V** times **V**; then divide by 10.

The code name for **V** times **V** is V^2.

So we can say the whole rule in a shorter form. Find V^2; then divide by 10.

We would write this:

$$S = \frac{V^2}{10}$$

where V is the speed in mph and S is the braking distance in feet.

The Falling Stone

The numbers 0, 1, 4, 9, 16, 25 . . ., as we saw earlier, are called the square numbers. The law for a falling stone is closely connected with these.

t the number of seconds a stone falls	**h** the number of feet the stone falls
0	**0**
1	**16**
2	**64**
3	**144**
4	**256**

The numbers written under **h** are found by taking the square numbers and multiplying them by 16, like this:

$$\begin{aligned} 0 \times 16 &= 0 \\ 1 \times 16 &= 16 \\ 4 \times 16 &= 64 \\ 9 \times 16 &= 144 \\ 16 \times 16 &= 256 \end{aligned}$$

The rule for finding **h** can be stated as follows:
See how many seconds the stone has fallen.
Multiply this number by itself.
Multiply the result by 16.

For example: How far does a stone fall in 5 seconds?
Here we begin with the number 5.
We multiply it by itself. This gives 25.
We multiply 25 by 16. This gives 400.
So, in five seconds, a stone falls 400 feet.

In shorthand: when $t = 5$, $h = 400$.

The rule could be written in shorthand: Find **t** times **t**; then multiply by 16.

The code name for **t** times **t** is t^2.

So we can say the rule in a shorter form. Find t^2; then multiply by 16.

This would be written:

$$h = 16\,t^2$$

This rule can be used to find the height of a cliff. Throw a stone from the top and see how many seconds it takes to strike the land below.

The Steel Bar

On page 19, an imaginary experiment with a steel bar was described.

The formula that connects **d** and **w** in the table on page 19 is:

$$w = 100{,}000\, d^2$$

For example, if $d = 0.3$, we have $d^2 = 0.3 \times 0.3 = 0.09$ Multiplying this by 100,000 gives $0.09 \times 100{,}000 = 9{,}000$. This agrees with the value of **w** given in the table.

The Use of Decimals. You will notice how decimals come into this question. In science and engineering we often have to deal with decimals. When we measure something, it is unlikely that it will give an exact whole number.

If you plan to become a scientist or an engineer, you should practice calculations with decimals until you are thoroughly at home with them. In particular, learn to *feel* how big (or how small) numbers are. A human hair, for instance, is about .003 inch thick. This helps you to imagine the size of the number .003.

A Mistake to Avoid

We have used 5^2 to stand for 5×5, and x^2 to stand for x times x. We call 5^2 the square of 5, and x^2 the square of x.

Whenever we multiply a number by itself, we say we are *squaring* it.

Examples:

Square 3	**Answer 9**
Square 5	**Answer 25**
Square 2	**Answer 4**
Square 8	**Answer 64**
Square 6	**Answer** ....
Square 4	**Answer** ....
Square 7	**Answer** ....
Square 10	**Answer** ....
Square 1	**Answer** ....

It is not quite clear what we should mean by

$$2 \times 3^2$$

Jack might reason like this: 3^2 means "the square of 3" which is 3×3, and that is 9. So 2×3^2 should mean 2×9. The answer is 18.

Bill might argue this way: 2×3 is 6. So 2×3^2 should mean 6^2. This is the square of 6. The answer is 36.

Both boys are using sensible arguments. In fact, they are doing the same things but in a different order.

Jack says,
Take the number 3.
Square it.
Then multiply by 2.

Bill says,
Take the number 3.
Multiply it by 2.
Square the result.

As we have seen, these lead to different answers. Because both arguments are so sensible, we have to make an agreement, so as to avoid misunderstandings. *All the mathematicians in the world have agreed to use Jack's way of looking at things.*

So you should read

2×3^2	as	$2 \times 9 = 18$
2×4^2	as	$2 \times 16 = 32$
3×5^2	as	$3 \times 25 = 75$
4×10^2	as	$4 \times$ .... $= 400$
5×10^2	as	$5 \times$ .... = ...
2×6^2	as	$2 \times$ .... $= 72$
10×6^2	as	$10 \times$ .... = ...
2×5^2	as	$2 \times$ .... $= 50$
4×5^2	as	$4 \times$ .... = ...
10×5^2	as	.... $\times$ .... = ...
4×3^2	as	.... $\times$ .... = ...
5×3^2	as	.... $\times$ .... = ...
2×10^2	as	.... $\times$ .... = ...

In the same way, $10x^2$ corresponds to the instructions:

Think of a number x.
Square it.
Multiply the result by 10.

If you like you can read $10x^2$ as "ten times the square of x."

A little earlier, we considered *The Falling Stone* and *The Steel Bar*. The calculations in these cases would seem wrong to you if you used Bill's way of looking at the equations $h = 16t^2$ and $w = 100{,}000d^2$. You should always look at such equations in the way Jack looks at them because that is what the people who write the equations expect you to do.

Can you fill in the tables below?

Table I $y = 3x^2$

x	y
0	--------
1	--------
2	--------
3	--------
4	--------
5	--------

Table II $y = 2x^2$

x	y
0	--------
1	--------
2	--------
3	--------
4	--------
5	--------

Table III $y = 10x^2$

x	y
0	--------
1	--------
2	--------
3	--------
4	--------
5	--------

Table IV $y = 5x^2$

x	y
0	--------
1	--------
2	--------
3	--------
4	--------
5	--------

A Way of Checking Your Answers

The answers in Table IV should be half the answers in Table III. Also, if you add together the answers in Tables I and II, you should get the answers in Table IV; that is, the first number in Table I added to the first number in Table II, should give the first number in Table III, and so on for the second, third, fourth, and fifth numbers.

How Far Can You See?

Suppose it is a clear day and you are at the seaside or in some place like Illinois where the ground is very flat. You are at the top of a building or in a plane. How many miles will you be able to see?

Or you can put the question the other way round and say, "If you want to see a town 30 miles away, how high must you be?"

Shorthand: Let **n** stand for the number of miles you want to see; **h** for the number of feet you have to climb above the surrounding flat country.

There is a rule about the relationship of **h** to **n**:

$$h = \tfrac{2}{3}n^2$$

In words:

Take the number of miles you want to see.
Square it.
Find two-thirds of the result.

Find the height from which you can see each of the following distances:

n	h
3	--------
6	--------
9	--------
30	--------
60	--------
90	--------

If, later on, you study geometry, you will learn the reason why this rule works.

The Force of the Wind

The way the wind presses against a building can be found from the formula

$$w = \frac{V^2}{400}$$

Here **v** stands for the speed of the wind in miles per hour, **w** stands for the weight in pounds that will press on each square foot of a building.

v	w
10	--------
20	--------
30	--------
40	--------
50	--------
60	--------
70	--------
80	--------
90	--------
100	--------

If you work out how many square feet there are on the side of a car, it will not surprise you that a high wind can sometimes blow a car right off the road.

Electric Heater

If you took an electric heater that was designed for 110 volts, and connected it to a 220 volt power supply, the current would be twice as large as the manufacturers intended, the rate at which heat was produced would be four times as big—that is, if the wire did not melt or a fuse blow. If you multiply the current by 3, the rate of heat generation is multiplied by 9.

Here again we have the square numbers, 1, 4, 9, and so on.

The Pull of the Earth

The square numbers also come into the theory of gravitation. At present you are about 4,000 miles from the center of the earth. The earth pulls you toward itself—that is why you stay on the floor. If the earth did not pull you, and you gave a little push with your legs, you would fly up and hit the ceiling. It is the pull of the earth that holds you down.

If you could get 8,000 miles away from the center of the earth—just twice as far away as you are now—the earth would still pull you, but its pull would not be so strong.

Some families have little weighing machines worked by springs. If a tower could be built 4,000 miles high, and you stood on such a weighing machine at the top of the tower, it would show that you had very much less weight. You would be twice as far from the earth's center as usual; your weight would be ¼ of its usual amount.

If you went to 3 times the usual distance, your weight would be 1/9 of the usual amount.

This table shows how distance and weight are connected. Can you fill in the gaps?

Distance	Weight
1	1
2	¼
3	1/9
4	1/16
5	------
6	------
7	------
8	------
------	1/81
------	1/100

We could say in shorthand that if you go **n** times as far away, your weight becomes $\frac{1}{n^2}$ of what it was before you went.

We say that gravity obeys *the inverse square law.* "Inverse" means that distance and weight vary in opposite directions—the bigger the distance, the less the weight. If you *multiply* the distance by n, you *divide* the weight by n^2.

The inverse square law is very common in physics. The same law holds for electric charges—if you double the distance between them, you divide the force by 4. The same law holds good in magnetism.

The inverse square law occurs also in the theory of light. Suppose an electric lamp produces a certain degree of brightness at the distance of one foot. At a distance of two feet, it would produce ¼ of that brightness, at 3 feet, only 1/9, at 4 feet, only 1/16, and so on.

The Widespread Square Law

At the beginning of this chapter, we mentioned fifteen different questions, all connected with the idea of $\mathbf{n}^2$. We have suggested the answers to all of these except No. 3 and No. 15.

No. 3 is very simple. To cover a square floor, 10 feet by 10 feet, we need 100 square feet of floor covering. $100 = 10 \times 10 = 10^2$. Many arithmetic problems deal with questions of this kind.

No. 15 we cannot go into detail. The idea behind it can be learned from a simple experiment. Suppose you have a cardboard box. Now if you make another box to twice the scale, you will find that you use 4 times as much cardboard, and that the new box holds 8 times as much.

In the same way, if you could take an animal and double its scale, you would multiply its weight by 8, but it seems that its strength would only be multiplied by 4. Thus the larger an animal gets, the less nimble it becomes. A flea enlarged to the scale of an elephant probably would be unable to jump at all.

We have by no means mentioned all the places where the Square Law occurs in science.

If you stretch a spring **n** inches, the energy stored in it is proportional to n^2. If the speed of a car is multiplied by n, the damage it can do is multiplied by n^2. If you throw a ball n times as fast as another person, it will travel n^2 times as far. If you multiply the thickness of a plank by n, you multiply the load it can support by n^2. There are countless other applications of the law.

There are, of course, many other laws that occur in nature and that you will meet as you go on with your studies.

In this book we have tried to show how algebra helps you to write down scientific laws. We have touched only the beginnings of algebra and the beginnings of science. If you wish to study science or engineering, or if you are interested in mathematics, you should learn more algebra and keep practicing it until you can think in algebra without any effort at all. You will find this a great help. And it can become very interesting, too.

Answers to Problems

Page 7

3. Subtract number from 1000.
4. 10, 7, ½.
   Multiply the number by ½, or divide it by 2.
5. Add 2 to the number.
6. Subtract 3 from the number.
7. Subtract the number from 10.
8. Multiply the number by 3.
9. Divide the number by 4.
10. Multiply the number by itself.

Page 8

A2. 0 2 6 12 20 30
2 4 6 8 10
2 2 2 2

A3. 0 3 8 15 24 35
3 5 7 9 11
2 2 2 2

A4. Last line: $5 \times 8 = 40$
0 4 10 18 28 40
4 6 8 10 12
2 2 2 2

A5. $4 \times 8 = 32$; $5 \times 9 = 45$.
0 5 12 21 32 45
5 7 9 11 13
2 2 2 2

Page 9

B1. All answers are 1. *B2.* All answers are 2.

B3. $5 \times 7 - 4 \times 8$
$6 \times 8 - 5 \times 9$
$7 \times 9 - 6 \times 10$
All answers are 3.

B4. All answers are 10.

B5. $6 \times 10 - 7 \times 8$
$7 \times 11 - 8 \times 9$
Answers are 0, 1, 2, 3, 4, 5.

B6. $7 \times 12 - 8 \times 10$
Answers are 0, 1, 2, 3, 4.

B7. $7 \times 8 - 5 \times 9$
$8 \times 9 - 6 \times 10$
Answers are 7, 8, 9, 10, 11, 12.

B8. $5 \times 7 - 4 \times 6$
$6 \times 8 - 5 \times 7$
Answers are 3, 5, 7, 9, 11, 13.

B9. $5 \times 5 - 4 \times 4$
$6 \times 6 - 5 \times 5$
Answers are 1, 3, 5, 7, 9, 11.

Pages 10 and 11

C1. The answer is always 2.
C2. Not a trick.

Page 11

C3. Not a trick.
C4. Answer is always 2.

Page 15

5. $x + 2$
6. $x - 3$
7. $10 - x$
8. $3x$
9. $\frac{1}{4}x$ or $\frac{x}{4}$

Page 16

F2. $y = x + 2$
F3. $y = x + 3$
F4. $y = x - 1$
F5. $y = 2x$

Page 17

F6. $y = 4x$
F7. $y = 10x$
F8. $y = 2x + 1$
F9. $y = 2x + 3$
F10. $y = 3x + 1$
F11. $y = 2x - 1$
F12. $y = 3x - 2$

Page 19

Steel Bar: 36,000; 49,000.
Car Braking: 250; 360.

Page 20

Rollers—$p = 2r$
Pulley—$h = 2a$
Weighing Machine—$n = 4w$

Page 22

2. Answers are 11, 4.
Law of Floating: $M = W$
Airplane Engines' Load: $P = W$

Page 23

Page 7. Nancy's rule.
Page 8. Arithmetic pattern A1.
Page 8. Arithmetic pattern B9.
Page 19. Steel Bar
Page 19. Car Braking.

Page 24

Cans of Water—Four times; 4, 9, 16.
Pennies—More than double; No; More.

Page 25

Pendulum—About the same; Longer; Multiply by 4. Multiply by 9.
Writing Numbers—$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$. 32. 128.

Page 26

81; $729 = 3 \times 3 \times 3 \times 3 \times 3 \times 3$;
$10 \times 10 = 10^2$; $7 \times 7 \times 7 = 7^3$;
$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6$;
$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$;
$5 \times 5 = 5^2$; $10 \times 10 \times 10 = 10^3$.
$2^3 = 8$; $5^2 = 25$; $7^2 = 49$; $2^5 = 32$
$10^3 = 1000$; $3^4 = 81$; $3^3 = 27$; $3^2 = 9$
$6^2 = 36$; $11^2 = 121$; $8^2 = 64$; $10^2 = 100$
$5^3 = 125$; $3^5 = 243$

Page 27

6^2; 7^2; 8×8; 9^2 81; 10^2; 10×10.

Page 28

36; 16; 49; 100; 1.
Problems at bottom of column 2:
100; $5 \times 100 = 500$; 36; $10 \times 36 = 360$;
25; $4 \times 25 = 100$; $10 \times 25 = 250$; $4 \times 9 = 36$;
$5 \times 9 = 45$; $2 \times 100 = 200$.

Page 29

Table I: 0, 3, 12, 27, 48, 75.
Table II: 0, 2, 8, 18, 32, 50.
Table III: 0, 10, 40, 90, 160, 250.
Table IV: 0, 5, 20, 45, 80, 125.
How far can you see?
$h = 6, 24, 54, 600, 2400, 5400$.
Force of the wind: $w = \frac{1}{4}, 1, 2\frac{1}{4}, 4, 6\frac{1}{4}, 9, 12\frac{1}{4}, 16, 20\frac{1}{4}, 25$.

Page 30

The Pull of the Earth: $\frac{1}{25}$; $\frac{1}{36}$; $\frac{1}{49}$; $\frac{1}{64}$; 9; 10.