MERF 1

Maxwell's Equations in Relativity Form.

N22

M 48

$$\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t} \qquad -\operatorname{curl} E = \frac{1}{c}\frac{\partial H}{\partial t}.$$

$x^2+y^2+z^2-c^2t^2$ is to be invariant.

Let $ict = l$ $\quad x^2+y^2+z^2+l^2$ invariant

so orthogonal transformations.

$$\frac{\partial E}{\partial l} = \frac{\partial E}{ic\,\partial t} = \frac{1}{i}\operatorname{curl} H$$

$$\therefore \quad \operatorname{curl} H = \frac{\partial\, iE}{\partial l} = \frac{\partial \mathcal{E}}{\partial l}$$

where $\mathcal{E} = iE$.

$$\frac{\partial H}{\partial l} = \frac{\partial H}{ic\,\partial t} = -\frac{\operatorname{curl} E}{i} = \operatorname{curl} iE = \operatorname{curl} \mathcal{E}.$$

$$\mathcal{E} = (U, V, W) \qquad H = (L, M, N).$$

I $\quad \operatorname{curl} H = \dfrac{\partial \mathcal{E}}{\partial x_4}$

$$\frac{\partial U}{\partial x_4} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}$$

$$\frac{\partial V}{\partial x_4} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}$$

$$\frac{\partial W}{\partial x_4} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

MERF 2 N21 M 49

We have

1) $0 = -\frac{\partial N}{\partial x_2} + \frac{\partial M}{\partial x_3} + \frac{\partial U}{\partial x_4}$

2) $0 = \frac{\partial N}{\partial x_1} - \frac{\partial L}{\partial x_3} + \frac{\partial V}{\partial x_4}$

3) $0 = -\frac{\partial M}{\partial x_1} + \frac{\partial L}{\partial x_2} + \frac{\partial W}{\partial x_4}$

$\cancel{(4)} =$

(4) $0 = -\frac{\partial U}{\partial x_4} - \frac{\partial V}{\partial x_2} - \frac{\partial W}{\partial x_3}$

(4) could be written without minus sign. Antisymmetry.

II. $\frac{\partial H}{\partial x_4} = \text{curl } E$

1) $0 = -\frac{\partial W}{\partial x_2} + \frac{\partial V}{\partial x_3} + \frac{\partial L}{\partial x_4}$

2) $0 = \frac{\partial W}{\partial x_1} - \frac{\partial U}{\partial x_3} + \frac{\partial M}{\partial x_4}$

3) $0 = -\frac{\partial V}{\partial x_1} + \frac{\partial U}{\partial x_2} + \frac{\partial N}{\partial x_4}$

4) $0 = -\frac{\partial L}{\partial x_1} - \frac{\partial M}{\partial x_2} - \frac{\partial N}{\partial x_3}$

In each set $\frac{\partial}{\partial x_1}(1) + \frac{\partial}{\partial x_2}(2) + \frac{\partial}{\partial x_3}(3) + \frac{\partial}{\partial x_4}(4) = 0.$

N20

M50

$$\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}=\frac{\partial P}{\partial Y}\sin\alpha$$

$$-\frac{1}{\sin\alpha}\frac{\partial Q}{\partial X}-\frac{\partial Q}{\partial Y}\cos\alpha-\frac{\partial P}{\partial X}\cos\alpha-\frac{\partial P}{\partial Y}c'\alpha$$

$$\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x} \qquad x = aX+bY, \quad y = cX+dY$$

$$xf+yg = XF+YG$$

$$\therefore\ (aX+bY)f+(cX+dY)g = XF+YG$$

$$af+cg = F \qquad f=\frac{dF-cG}{ad-bc}$$

$$bf+dg = G. \qquad g=\frac{-bF+aG}{ad-bc}$$

$$\frac{\partial}{\partial X}= a\frac{\partial}{\partial x}+c\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial Y}= b\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}$$

$$\frac{\partial P}{\partial Y}-\frac{\partial Q}{\partial X}=b\frac{\partial P}{\partial x}+d\frac{\partial P}{\partial y}-a\frac{\partial Q}{\partial x}-c\frac{\partial Q}{\partial y}$$

$$=\left[b\frac{\partial}{\partial x}+d\frac{\partial}{\partial y}\right](ap+cq)-\left(a\frac{\partial}{\partial x}+c\frac{\partial}{\partial y}\right)(bp+dq)$$

$$= ab\frac{\partial p}{\partial x}+ad\frac{\partial p}{\partial y}+bc\frac{\partial q}{\partial x}+cd\frac{\partial q}{\partial y}$$

$$-ab\frac{\partial p}{\partial x}-bc\frac{\partial p}{\partial y}-ad\frac{\partial q}{\partial x}-cd\frac{\partial q}{\partial y}$$

$$=(ad-bc)\left(\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}\right)$$

We try to make an extensive list of expressions that give tensors, i.e expressions with a prescribed transformation rule, which ensures that equations in any co-ordinate system have the same form.

1) Let φ be a scalar, and let $u_i = \dfrac{\partial \varphi}{\partial x^i}$.

If we introduce new coordinates, as usual, by the equation $x^i = t^i_j X^j$ we have

$$\frac{\partial \varphi}{\partial X^j} = \frac{\partial \varphi}{\partial x^i}\frac{\partial x^i}{\partial X^j} = \frac{\partial \varphi}{\partial x^i} t^i_j$$

Thus $U_j = t^i_j u_i$. This is the transformation rule, given in equation (4), for a covariant vector, so u_i with lower index is justified.

2)(i) Let v^i be a vector and $a^i_j = \dfrac{\partial v^i}{\partial x^j}$

$$\frac{\partial v^i}{\partial X^j} = \frac{\partial v^i}{\partial x^s}\frac{\partial x^s}{\partial X^j} = a^i_s t^s_j$$

Also $\dfrac{\partial v^i}{\partial X^j} = \dfrac{\partial}{\partial X^j}(t^i_k V^k) = t^i_k \dfrac{\partial V^k}{\partial X^j} = t^i_k A^k_j$

So $a^i_s t^s_j = t^i_k A^k_j$

The change k to i has the form of $u^i = t^i_k U^k$.

From j to s the form of $u_s t^s_j = U_j$

Thus a^i_s is a tensor with contravariant i and co-variant s. Any equation consistent with this index system will keep its form in all systems

N18

M
52

2(ii) The result is slightly clearer for covariant V_i.

Let $a_{ij} = \dfrac{\partial v_i}{\partial x^j}$

Then $A_{rs} = \dfrac{\partial V_r}{\partial X^s} = \dfrac{\partial V_r}{\partial x^j}\dfrac{\partial x^j}{\partial X^s} = \dfrac{\partial V_r}{\partial x^j} t^j{}_s)$

$$= t^j{}_s \frac{\partial}{\partial x^j}\left(t^i{}_r v_i\right)$$

$$= t^i{}_r t^j{}_s \frac{\partial v_i}{\partial x^j}$$

$$= t^i{}_r t^j{}_s \, a_{ij} .$$

Thus a_{ij} is doubly covariant tensor.

3. $K_{ij} = \dfrac{\partial v_i}{\partial x_j} - \dfrac{\partial v_j}{\partial x_i}$ is a covariant tensor

N17

M 53

$L\ M\ N = \text{curl}\ \mathcal{A}$

$$\begin{matrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 \end{matrix}$$

$$L = \frac{\partial \mathcal{A}_3}{\partial x_2} - \frac{\partial \mathcal{A}_2}{\partial x_3} =$$

$$M = \frac{\partial \mathcal{A}_1}{\partial x_3} - \frac{\partial \mathcal{A}_3}{\partial x_1} = \lambda_{13}.$$

Maxwell's equations with right handed axes. (N16)

M 54

$$\text{curl } H = \frac{1}{c}\frac{\partial E}{\partial t} \qquad \begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ L & M & N \end{array}$$

$$\frac{1}{c}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \quad \text{etc}$$

change of sign from Klein

$$l = ict \qquad U = iX$$

$$\frac{\partial U}{\partial l} = \frac{\partial . iX}{\partial ict} = \frac{1}{c}\frac{\partial X}{\partial t} = N_y - M_z$$

$$[I'] \left\{ \begin{aligned} 0 &= - N_y + M_z + U_l \\ 0 &= N_x - L_z + V_l \\ 0 &= -M_x + L_y + W_l \\ 0 &= -U_x - V_y - W_z . \end{aligned} \right.$$

If ~~$U = \lambda_{14}$ $V = \lambda_{24}$ $W = \lambda_{34}$~~
~~$L = \lambda_{23}$ $M = \lambda_{31}$ $N = \lambda_{12}$~~

$$0 = -\frac{\partial \lambda_{12}}{\partial x_2} + \frac{\partial \lambda_{31}}{\partial x_3} + \frac{\partial \lambda_{14}}{\partial x_4}$$

i.e. $0 = -\frac{\partial \lambda_{12}}{\partial x_2} - \frac{\partial \lambda_{13}}{\partial x_3} - \partial$

First eqn $0 = -\frac{\partial N}{\partial x_2} + \frac{\partial M}{\partial x_3} + \frac{\partial U}{\partial x_4}$

is to be $0 = \frac{\partial \lambda_{12}}{\partial x_2} + \frac{\partial \lambda_{13}}{\partial x_3} + \frac{\partial \lambda_{14}}{\partial x_4}$

$U = \lambda_{14}$ $N = \lambda_{21}$ $M = \lambda_{13}$
a reversal of indices in LMN.

N 15

M 55

Let $c\,dt = i\,dx_4$ — no. Take $x_4 = ict$

$$E = -\frac{1}{c}\frac{\partial \mathcal{A}}{\partial t} - \operatorname{grad} V$$

$$= -\frac{\partial \mathcal{A}}{i\,\partial x_4} - \operatorname{grad} V$$

$$= i\frac{\partial \mathcal{A}}{\partial x_4} - \operatorname{grad} V$$

~~If $V = i\varphi$~~

$$(UVW) = iE = -\frac{\partial \mathcal{A}}{\partial x_4} - i\operatorname{grad} V$$

If $V = i\varphi$

$$\mathcal{E} = (UVW) = -\frac{\partial \mathcal{A}}{\partial x_4} - \operatorname{grad}\varphi .$$

$$H = \operatorname{curl} \mathcal{A}$$

Pointwise limit of measurable fns is measurable fn p39

p56 $f_n(x) \to f(x)$ $|f_n(x)| \le \varphi(x)$ φ integrable

$$\int f_n(x) \to \int f(x)$$

(N14)

m 56

p 60 XYZ electric LMN magnetic

1) $\frac{1}{c}\frac{\partial X}{\partial t} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y}$ $\quad -\frac{1}{c}\frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$

2) $\frac{1}{c}\frac{\partial Y}{\partial t} = \frac{\partial N}{\partial x} - \frac{\partial L}{\partial z}$ $\quad -\frac{1}{c}\frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}$

3) $\frac{1}{c}\frac{\partial Z}{\partial t} = \frac{\partial L}{\partial y} - \frac{\partial M}{\partial x}$ $\quad -\frac{1}{c}\frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}$

4) $0 = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$ $\quad 0 = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}.$

Here $\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3) = \frac{1}{c}\frac{\partial}{\partial t}(4).$

The equations are $\frac{\partial}{\partial x_j} F_{ij} = 0$

$\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j} F_{ij}\right]$ has sym. diff, antisym F_{ij}
has symmetric differentiation, antisymmetric F_{ij}.

(Nr13) (1)

M 57

Klein C XIX. Chap 2. B.

§1 Maxwell equations for free ether.

Heaviside put Maxwell eqns in vector form

a) $\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t}$ b) $\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}$

c) $\operatorname{div} E = 0$ d) $\operatorname{div} H = 0$.

Herz used XYZ for electric force; LMN magnetic.

$$\mathrm{I}\begin{cases} \text{(i)}\ \frac{1}{c}\frac{\partial X}{\partial t} = \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y} \\ \text{(ii)}\ \frac{1}{c}\frac{\partial Y}{\partial t} = \frac{\partial N}{\partial x} - \frac{\partial L}{\partial z} \\ \text{(iii)}\ \frac{1}{c}\frac{\partial Z}{\partial t} = \frac{\partial L}{\partial y} - \frac{\partial M}{\partial x} \\ 0 = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\ \text{(iv)} \end{cases} \qquad \mathrm{II}\begin{cases} -\frac{1}{c}\frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \\ -\frac{1}{c}\frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \\ -\frac{1}{c}\frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \\ 0 = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} \end{cases}$$

The four rows are related by

(1) $\frac{\partial}{\partial x}(\text{i}) + \frac{\partial}{\partial y}(\text{ii}) + \frac{\partial}{\partial z}(\text{iii}) = \frac{\partial}{\partial t}(\text{iv})$ in I and II

Let $\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. Then

(3) $\square X = 0 \quad \square Y = 0 \quad \square Z = 0 \quad \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0$

(4) $\square L = 0 \quad \square M = 0 \quad \square N = 0 \quad \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0$

Are there linear substitutions of $x\,y\,z\,t$ and $XYZLMN$ that leave our system of equations invariant?

The operator $\square$ is left invariant by any transformation that leaves $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$ invariant under homogeneous linear transformation. (We take absolute invariance, not multiplied by constant.)

To show directly how Maxwell eqns can be kept invariant would require very obscure calculations. We follow Minkowski, who pointed out a hidden symmetry.

(N 12)

M 58

Klein. Vol 2. p 59.

§1 $\frac{1}{c}\frac{\partial E}{\partial t} = \text{curl } H \qquad \frac{1}{c}\frac{\partial H}{\partial t} = -\text{curl } E$

$\text{div } E = 0 \qquad \text{div } H = 0.$

XYZ electric force. LMN magnetic.

He uses left axes.

1) $\frac{1}{c}\frac{\partial X}{\partial t} = M_z - N_y \qquad -\frac{1}{c}\frac{\partial L}{\partial t} = Y_z - Z_y$

2) $\frac{1}{c}\frac{\partial Y}{\partial t} = N_x - L_z \qquad -\frac{1}{c}\frac{\partial M}{\partial t} = Z_x - X_z$

3) $\frac{1}{c}\frac{\partial Z}{\partial t} = L_y - M_x \qquad -\frac{1}{c}\frac{\partial N}{\partial t} = X_y - Y_x$

4) $0 = X_x + Y_y + Z_z \qquad 0 = L_x + M_y + N_z$

Relation between eq $\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3) = \frac{1}{c}\frac{\partial(4)}{\partial t}$

§2 Let $l = ict$

$iX = U \qquad iY = V \qquad \cancel{iW = Z}\ iZ = W$

1') $0 = \ .\ + N_y - M_z + U_l \qquad 0 = \ .\ W_y - V_z + L_l$

2') $0 = -N_x\ .\ L_z + V_l \qquad 0 = -W_x + U_z + M_l$

3') $0 = M_x - L_y\ .\ + W_l \qquad 0 = V_x - U_y\ .\ + N_l$

4') $0 = -U_x - V_y - W_z \qquad 0 = -L_x - M_y - N_z$

$\frac{\partial}{\partial x}(1') + \frac{\partial}{\partial y}(2') + \frac{\partial}{\partial z}(3') + \frac{\partial(4')}{\partial l} = 0$

N41

M59

The 4 vector.

Let $x_4 = ct$.

If a region V_0 moves with velocity v, its volume is reduced to $V_0\sqrt{1-\frac{v^2}{c^2}}$

Charge is invariant.

$$\therefore \quad \rho V = \rho_0 V_0$$

$$\therefore \quad \rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$ds^2 = -(dx_1^2 + dx_2^2 + dx_3^2) + c^2dt^2$$

$$= dt^2(c^2 - v^2)$$

$$\therefore \frac{ds}{dt} = \sqrt{c^2-v^2} = c\sqrt{1-\frac{v^2}{c^2}}$$

$\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}, \frac{dx_4}{ds}$ is a 4 vector.

$$\text{i.e.} = \frac{dx_1}{dt}\Big/\frac{ds}{dt}, \ \ldots, \ \frac{dx_4}{dt}\Big/\frac{ds}{dt}$$

$$= \frac{v_1}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_2}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_3}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{c}{c\sqrt{1-\frac{v^2}{c^2}}}$$

~~So $\frac{v_1}{c\sqrt{1-\frac{v^2}{c^2}}}, \frac{v_2}{c\sqrt{\;}}, \frac{v_3}{c\sqrt{\;}}, \frac{c}{c}$~~

is a 4 vector. Multiply by ρ_0

$\frac{\rho v_1}{c}, \frac{\rho v_2}{c}, \frac{\rho v_3}{c}, \rho$ is 4 vector.

i.e. $\frac{j_1}{c}, \frac{j_2}{c}, \frac{j_3}{c}, \rho$ is.

N10

M 60

Maxwell's Equations.

$$(\mathrm{I}) \ldots \operatorname{curl} H = \frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right) \qquad \operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t} \ldots (\mathrm{II})$$

Let $H = \operatorname{curl} A$. (I) becomes

$$\frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right) = \operatorname{curl}\operatorname{curl} A = \operatorname{grad}\operatorname{div} A - \nabla^2 A.$$

$$(\mathrm{III})\ E = -\frac{1}{c}\frac{\partial A}{\partial t} - \operatorname{grad} V \ *$$

V electric potential.

$$\frac{4\pi j}{c} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \frac{1}{c}\operatorname{grad}\frac{\partial V}{\partial t} = \operatorname{grad}\operatorname{div} A - \nabla^2 A$$

$$\therefore\ \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \frac{4\pi j}{c} = \operatorname{grad}\left[\operatorname{div} A + \frac{1}{c}\frac{\partial V}{\partial t}\right]$$

Now A is not completely determined: it is defined by $\operatorname{curl} A = H$. We fix it by the further condition $\operatorname{div} A + \frac{1}{c}\frac{\partial V}{\partial t} = 0$. (IV)

Then
$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi j}{c}.$$

From III
$$\operatorname{div}\operatorname{grad} V = -\operatorname{div} E - \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div} A$$
$$= -\operatorname{div} E + \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial V}{\partial t}$$
$$\therefore\ \nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\operatorname{div} E = -4\pi\rho.$$

* This equation discussed in N 31.

(N9) m 61

$$\operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}$$

$$H = \operatorname{curl} A$$

$$\operatorname{curl} E = -\frac{1}{c}\operatorname{curl}\frac{\partial A}{\partial t}$$

$$\operatorname{curl}\left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} \mp \operatorname{grad}\varphi$$

$$\operatorname{curl}\operatorname{curl} H = \frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl} E = -\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2}$$

$$\operatorname{curl}\operatorname{curl} H = -\nabla^2 H + \frac{\partial}{\partial x}\operatorname{div} H$$

$$\operatorname{div} H = 0. \text{ so } \nabla^2 H - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0.$$

$$\text{so } \nabla^2 \operatorname{curl} A - \frac{1}{c^2}\operatorname{curl}\frac{\partial^2 A}{\partial t^2} = 0$$

$$\text{consistent with } \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0$$

(N8) DC1 M62

Displacement current.

Ampère's Theorem states that the magnetic field produced by a current i in a loop is the same as that of a magnetic shell, of strength i, bounded by the loop.

Consequences (1) the magnetic potential at a point P is $i\omega$, where ω is the solid angle subtended at P by the loop. (2) the work done on unit magnetic pole as it goes round a circuit threading the loop is $4\pi i$.

If we take any surface S bounded by a curve C, then C threads any circuit that crosses S.

Thus work done in a circuit of C is 4π times the current crossing S. If $\underset{\sim}{j}$ is the current density, this is $\iint_S \underset{\sim}{j}\, d\underline{S}$.

The work done in the circuit of C is

$$\int_C \underset{\sim}{H}\, ds = \iint_S \operatorname{curl} H\, dS$$

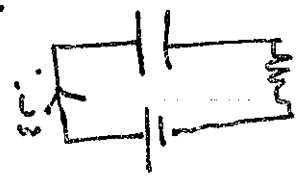
$$\text{Thus} \quad \operatorname{curl} H = 4\pi \underset{\sim}{j}. \qquad (I)$$

Now div curl H is identically zero, so this equation can only be valid if div $j = 0$., that is, if ~~current~~ charge neither accumulates in nor disappears from any region. This is normally the case for currents in a conductor. [Jeans. §534]

~~A problem arises~~

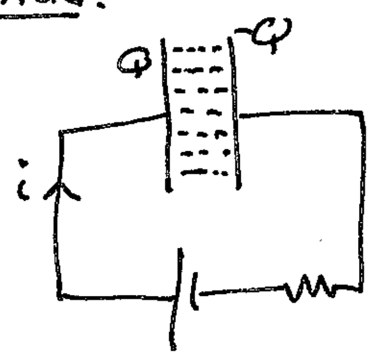
The condition div = 0 above is equivalent to saying that it does not matter which circuit bounded by C we choose for S.

A problem arises if the current ~~is~~ ~~a moving electron.~~ is not in a closed ~~circuit~~ loop, for instance in a ~~batter~~



N7 DC2 M 63

Maxwell, when working to embody the previously known experimental results in his system of differential equations, was led to the idea of <u>displacement current</u>.



Let Q be the charge on one ~~set of~~ plate of the capacitor. We suppose the plates large enough to make the field between them practically uniform. Area of each plate $= A$.

$E =$ field between them.

By Gauss Theorem, $EA = 4\pi Q$

Also $\dfrac{dQ}{dt} = i$. Hence $A\dfrac{dE}{dt} = 4\pi i$

Thus $i = A \cdot \dfrac{1}{4\pi}\dfrac{dE}{dt}$.

Thus we see that the changing electrical field, so to speak, replaces the effect of a current density ($\underset{\sim}{j}$) of magnitude $\dfrac{1}{4\pi}\dfrac{\partial E}{\partial t}$.

Accordingly, where both currents and changing electric fields occur, we replace j by $\underset{\sim}{j} + \dfrac{1}{4\pi}\dfrac{\partial E}{\partial t}$.

$$\text{curl } H = 4\pi\left(\underset{\sim}{j} + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right)$$

is free from objection, as $\text{div}\left(\underset{\sim}{j} + \dfrac{1}{4\pi}\dfrac{\partial E}{\partial t}\right) = 0$.

For we have $\text{div } E = 4\pi\rho$ so

$\dfrac{1}{4\pi}\dfrac{\partial}{\partial t}\text{div } E = \dfrac{\partial \rho}{\partial t}$. Now $\text{div } j$ measures the

(N6) DC3 M 64

rate (per unit volume) at which charge escapes from a small region, ~~so~~ so $\operatorname{div} j = -\partial\rho/\partial t$, and the sum is zero, as required.

Maxwell accordingly amended equation (I) to

$$\operatorname{curl} H = 4\pi\left(j + \frac{1}{4\pi}\frac{\partial E}{\partial t}\right) = 4\pi j + \frac{\partial E}{\partial t}.$$

Here j and E are in E.M.U. For ESU on RHS

$$\operatorname{curl} H = \frac{1}{c}\left(4\pi j + \frac{\partial E}{\partial t}\right).$$

If the current j is due to moving charges, $j = \rho v$ where v is the velocity of the charge $\rho\, dx\, dy\, dz$ at (x, y, z).

Jeans. p 393. (NS) $\frac{M}{65}$

Magnetic induction $= B = (a,b,c)$ has $\operatorname{div} B = 0$.

$\therefore \exists A :- \quad B = \operatorname{curl} A \qquad A = (F\ G\ H)$

A vector potential.

$\underline{f} = -A/c^2$

$(X\ Y\ Z) = E$

p 473 $\qquad -\frac{dB}{dt} = \operatorname{curl} E$

p 474

$E = -\frac{dA}{dt} - \operatorname{grad} \phi$

(N4) M/66

Ampère's Theorem. A current i in a loop produces the same magnetic field as a shell of strength $\mathbf{i}$ bounded by the loop.

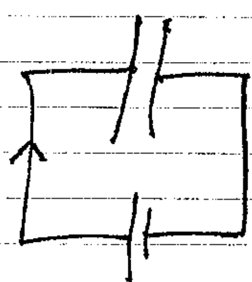
Consequences (1) magnetic potential at point P is ~~4π~~ iW where W is the solid angle subtended at P by the loop
(2) work done ~~by~~ on unit magnetic pole in a circuit that threads the current path is $4\pi \mathbf{i}$.

$$\int \mathbf{H}\, ds = \iint 4\pi \mathbf{j} \cdot dS$$

$$= \iint \operatorname{curl} \mathbf{H}\, dS \qquad \operatorname{curl} \mathbf{H} = 4\pi \mathbf{j}$$

The concept of threading a current is clear when the current is in a closed loop ~~but~~ ~~should go~~ but suppose current is because an electron has moved from A to B. We shall get different results, according to whether we consider a surface cutting AB or not.

Maxwell introduced the idea of a displacement current that in effect completes the circuit. Consider a circuit with a capacitor, and current i.



If there is uniform charge density σ on the plate, by Gauss Theorem $E = 4\pi\sigma$ gives the electric field between the plate.
If plates have unit area, charge is $4\pi\sigma$ and $4\pi\, d\sigma/dt = i$. Thus $i = dE/dt$.
Maxwell interpreted this as meaning that

p 566 $\text{current} = j + \frac{1}{4\pi}\frac{\partial E}{\partial t}$

$\operatorname{div} = \operatorname{div} j + \frac{1}{4\pi}\frac{\partial}{\partial t}(\operatorname{div} E) = -\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial t}.$